A stochastic model for the sustainable investment policy in a defined benefit pension fund

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Abstract

In this paper we study the problem of optimal asset allocation in a defined benefit pension fund that accumulates resources for pension benefit payments and is allowed to invest its capital in a security market with n risky assets. The pension fund operates in continuous time and is subject to financial and actuarial risks. We assume that the investment strategy takes expressly into account the sustainability of the fund, i.e. the balance between the active and retired members. We are, therefore, dealing with the case of a fund manager who wishes to maximise his utility depending on an empirical indicator. This indicator measures the financial sustainability of the pension fund and is expressed as the ratio of the fund wealth to a multiple of the current expenditure for pensions. The above problem is equivalent to a finite time horizon control problem which can be solved by using the dynamic programming approach. To this end, we will first obtain the so-called Hamilton-Jacobi-Bellman equation and then derive closed-form expressions for the optimal investment policy. Finally, we illustrate our model using data from a real Italian pension fund.

Keywords: Pension fund management, Optimal dynamic asset allocation, Sustainability ratio

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1 Introduction

The key objective of pension plans is to deliver retirement benefits, typically payable for life or over a set period, to the specified group of recipients. Managing such funds therefore entails constantly monitoring the risk exposure and regularly rebalancing the assets. This paper is directly related to these topics and proposes a method (mainly based on stochastic optimal control theory) to determine the optimal investment policy for a pension fund's wealth, given certain financial and actuarial risks.

We consider the case of a retirement scheme operating in continuous time whose wealth is invested in a number of risky assets. The fund value dynamic is basically determined by three single components: the global investment returns and the contribution cash flows on the asset side and the outgoing benefits on the liability side. We assume that the asset manager has to choose the optimal asset allocation at any time in order to maximise the fund value under the constraint driven by a risk indicator that aims to assess the sustainability of the fund. By sustainability we mean the pension fund's ability over time to match the liabilities promised to plan members through current and future assets.

This is, in fact, the situation of a specific type of retirement programmes operating in Italy - the Italian Professional Order Pension Funds. These are private closed schemes and operate according to a defined benefit plan with the aim to provide first-pillar pensions. Until 1995, Italian retirement funds for Professional Orders were administered directly by the State which would intervene if the funds becomes insolvent. Since 1995, this State sponsorship no longer exists: such retirement pensions are now provided by purely private pension plans.

The financial basis for the benefits promised to plan participants relies on a funding system in which the plan itself undertakes to invest employees' contributions. Nevertheless, this financing mechanism is not strictly speaking fully funded. It is, in fact, a mixed funded/PAYG pension plan (called ripartizione mista in the Italian pension system). The Italian regulator for Professional Orders retirement schemes is concerned with monitoring the financial self-sufficiency of the funds and, in particular, the financial sustainability established by the wealth management and by the portfolio investment strategy.

This paper is concerned with deriving optimal asset allocation patterns in partially funded pension schemes for workers enrolled in a Professional Order in Italy.

Recent years have seen the emergence of a vast literature on optimal asset allocation in pension funds. Security allocation problems were originally studied by Merton (1969, 1971) [13], [14]. Since these seminal papers, optimal consumption-investment modelling has become a central topic for financial mathematicians. However, most analyses of pension fund asset allocation assume the system is fully funded. Examples of such analysis can be found in
To the best of our knowledge, very few papers have dealt with mixed financing mechanisms in pension plans. Menoncin (2005) [12] tackles the asset allocation problem for a retirement programme which follows a PAYG rule and periodically revises its investment strategies. He assumes that both the total number of contributors and pensioners are random variables that can be covered in a complete financial market. There is thus no non-hedgeable risk, and closed-form solutions can be found for the optimal portfolio.

The model we use in this paper is based on a previous survey carried out by Hainaut and Devolder (2006) [8] that deals with optimal investment strategy in a funded pension plan. Their survey takes the case of a fund manager who wishes to maximise his utility function depending on a solvency ratio, defined as the market value of assets divided by the mathematical reserve, under a value-at-risk constraint.

Indeed, in this paper we consider that the retirement fund’s asset manager cares about a sustainability ratio that acts as the state variable in our model. This stochastic variable is introduced to take into account the Italian legislation indicator for retirement funds of the Professional Orders, i.e. the ratio of the fund value to a multiple of the current expenditure for pensions.

This paper is organised as follows. In Section 2, we first derive the stochastic differential equation for the sustainability ratio, which is the state process of our model. Afterwards, we construct the value function that the fund manager aims to continuously maximise through (optimal) asset allocation. In Section 3, we write down the Hamilton-Jacobi-Bellman equation and prove the verification theorem, which ensures that the solution to the above mentioned equation is exactly the value function defined in Section 2. The control policy thus achieved is the optimal dynamic asset allocation for the pension fund. Section 4 illustrates the procedure described in the previous section using a concrete example with real data taken from an Italian Professional Order Pension Fund. Section 5 concludes the paper.

2 Problem formulation

2.1 The financial market

Let us consider a continuous time, complete and frictionless financial market consisting of $n$ ($n > 1$) different risky assets.

At any time $t \in [0, T]$, where $T$ is the length of the fund manager’s investment horizon, the risky assets

$$F_{1t}, F_{2t}, \ldots, F_{it}, \ldots, F_{nt} \quad 1 \leq i \leq n$$
are driven by geometric Brownian motions

\[ dF_t = m_i F_t dt + \sum_{j=1}^{n} \sigma_{ij} F_t dW^F_{jt} \tag{2.1} \]

where \( F_0, m_i, \sigma_{ij} \in \mathbb{R}_+ \). Besides, \((W^F_{jt})_{t \geq 0}\) is a \((n)\)-dimensional vector Brownian motion defined on a probability space \((\Omega^F, \mathcal{F}^F, P^F)\) where \((\mathcal{F}^F_t)_{t \geq 0}\) is the filtration generated by \((W^F_t)_{t \geq 0}\) and \(P^F\) is the historical probability measure.

### 2.2 Contributions income and benefits payout

We assume that the stream of contributions \(\Gamma_t\) and of benefits \(B_t\) are stochastic and modelled by Ito processes. We recall that contributions and benefits are respectively positive and negative cash flows. In particular, equations (2.2) and (2.3) below represent the dynamics of the payment processes related to, respectively, the contributions from and benefits payed to all living members at a certain time \(t\)

\[ d\Gamma = \mu_\Gamma(t) \Gamma_t dt + \sigma_\Gamma(t) \Gamma_t dW^L_t \tag{2.2} \]

\[ dB = \mu_B(t) B_t dt + \sigma_B(t) B_t dW^L_t \tag{2.3} \]

where \(\mu_\Gamma(t), \mu_B(t), \sigma_\Gamma(t)\) and \(\sigma_B(t)\) are deterministic functions, \((W^L_t)_{t \geq 0}\) is a Brownian motion defined on the probability space \((\Omega^L, \mathcal{F}^L, P^L)\) that is independent of \((W^F_t)\) and we let \(\mathcal{F}^L_t\) be the filtration generated by \((W^L_t)_{t \geq 0}\).

The \(\mathcal{F}^L_t\)-adapted one-dimensional consumption process is the net payment stream given by \(B_t - \Gamma_t\).

### 2.3 The managed wealth and the sustainability ratio

Given the \(n\)-dimensional price process \(\{F_t\}_{t \geq 0}\) generated by (2.1), a Markovian relative portfolio strategy is any \(n\)-dimensional process \(\{\Pi_t\}_{t \geq 0}\) of the form \(\Pi_t = \Pi(t, F_t)\) for some function \(\Pi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n\).

In our specific case, the portfolio strategy has to satisfy a balance constraint. In particular, we assume that in a pension fund borrowing is prohibited, i.e., if \(\pi^i_t\) denotes the proportion of pension fund wealth invested in the \(i^{th}\) risky asset at time \(t\), we assume that the following constraint holds for \(t \in [0, T]\) and for \(i = 1, \ldots, n\):

\[ \pi^n_t - 1 = -\sum_{i=1}^{n-1} \pi^i_t \tag{2.4} \]

We denote by \(\mathcal{U} \subseteq \mathbb{R}^n\) the control space, i.e. the set of investment polices that satisfy the above mentioned constraint.
\[ U = \{ \Pi = (\pi_1, \ldots, \pi_i, \ldots, \pi_n) : \pi_n^t = 1 - \sum_{i=1}^{n-1} \pi_i^t \} \]

We require that \( \Pi_t \in U \) for \( t \in [0, T] \).

The fund wealth \( A \) is the value of a self-financing portfolio-consumption process, whose stochastic differential equation is

\[ dA_t = (m^\top \cdot \Pi_t A_t + \Gamma_t - B_t) dt + A_t \Pi_t^\top \sigma dW_t^F \quad (2.5) \]

where \( m \) is the \( n \)-dimensional vector containing expected returns of the risky assets and \( \sigma \) is the \((n, n)\) matrix of \( \sigma_{ij} \) obtained by the Choleski’s decomposition of the variance-covariance matrix of the assets \( \Sigma = \sigma^\top \sigma \). Superscript \( \top \) denotes the transpose of any vector or matrix.

We assume that the fund manager wants to define an investment strategy consistent with the goal of ensuring the sustainability of the fund. To achieve this, we introduce the sustainability ratio, i.e. the market value of the fund’s wealth divided by a reserve which is a multiple of the current expenditure for pensions. This ratio is an efficient and concise way of measuring the available surplus, and is recognised by Italian regulators as a good indicator of sustainability. The sustainability of the fund can thus be expressed by a diffusion process.

Let us define the sustainability reserve of the fund as

\[ R_t = \alpha \cdot B_t \quad (2.6) \]

where \( \alpha \in \mathbb{R}^+ \). By applying Ito’s lemma to (2.6), we obtain the dynamic of \( R_t \)

\[ dR_t = \mu_B(t)R_t dt + \sigma_B(t)R_t dW_t^L \quad (2.7) \]

Let us define the sustainability ratio \( S_t \) as

\[ S_t = \frac{A_t}{R_t} \quad (2.8) \]

**Proposition 2.1.** For all \( t \in [0, T] \), the process \( S_t \) verifies the stochastic differential equation

\[ dS_t = \left( (m^\top \cdot \Pi_t - \mu_B(t) + \sigma_B(t)^2) S_t + \frac{1}{\alpha} (\Psi_t - 1) \right) dt \]

\[ + (\Pi_t^\top \sigma dW_t^F - \sigma_B(t) dW_t^L) S_t \quad (2.9) \]

where \( \Psi_t = \frac{\Gamma_t}{B_t} \) is a stochastic function.

**Proof.** We obtain the dynamic of the sustainability ratio \( S_t \) as follows

\[ dS_t = d\frac{A_t}{R_t} = A_t \cdot d\left( \frac{1}{R_t} \right) + \frac{1}{R_t} \cdot dA_t \quad (2.10) \]

where, by applying Ito’s lemma
\[ d \left( \frac{1}{R_t} \right) = -\frac{1}{R_t} \left( (\mu_B(t) - \sigma_B(t)^2) \, dt + \sigma_B(t) dW_t^L \right) \quad (2.11) \]

Replacing (2.5) and (2.11) in (2.10), we obtain

\[
dS_t = \left( (m^T \cdot \Pi - \mu_B(t) + \sigma_B(t)^2) \, S_t + \frac{1}{\alpha} (\Psi_t - 1) \right) \, dt + \left( \Pi_t^T \cdot \sigma dW_t^F - \sigma_B(t) dW_t^L \right) S_t \quad (2.12)\]

Ito’s Lemma applied to \( \Psi_t \) gives

\[
d\Psi_t = \Psi_t \left( \mu_G(t) - \mu_B(t) + \sigma_G(t)^2 - \sigma_G(t) \sigma_B(t) \right) \, dt + (\sigma_G(t) - \sigma_B(t)) \, \Psi_t dW_t^L \quad (2.13)\]

The managed pension fund can be represented by the following stochastic differential equations with initial values \( S_0, \Psi_0 \) and an investment policy \( \Pi \).

\[
\begin{aligned}
    dS_t &= \left( (m^T \cdot \Pi + \mu_B(t) + \sigma_B(t)^2) \, S_t + \frac{1}{\alpha} (\Psi_t - 1) \right) \, dt \\
    &\quad + \sigma \cdot \Pi_t^T S_t dW_t^F - \sigma_B(t) S_t dW_t^L \\
    d\Psi_t &= \Psi_t \left( \mu_G(t) - \mu_B(t) + \sigma_G(t)^2 - \sigma_G(t) \sigma_B(t) \right) \, dt \\
    &\quad + (\sigma_G(t) - \sigma_B(t)) \, \Psi_t dW_t^L \\
    S(0) &= S_0 \\
    \Psi(0) &= \Psi_0 
\end{aligned} \quad (2.14)\]

The pair \( X := (S, \Psi) \) must therefore be considered as the state process.

### 2.4 The value function

We assume that the fund manager seeks to continuously maximise the utility arising from the sustainability ratio, by using an adapted investment policy, which is the control variable of our model.

**Definition 2.1.** A control law \( \Pi \) is called admissible if:

- \( \Pi(t, F_t) \in \mathcal{U} \) for all \( t > 0 \) and all \( F_t \in \mathbb{R}^n \);
- the SDE (2.12) has a unique solution corresponding to \( \Pi(t, F_t) \)

The class of admissible control laws is denoted by \( \Pi \).
The value function of our problem $V : \mathbb{R}_+ \times \mathbb{R}^2 \times \Pi \to \mathbb{R}$ is defined by

$$V(t, X, \Pi) = \mathbb{E} \left[ \int_0^T U(t, X_t^\Pi, \Pi_t) dt + K(X_T^\Pi) \right]$$

where $U : \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^n \to \mathbb{R}$ is the utility function and $K : \mathbb{R}^2 \to \mathbb{R}$ is the bequest function. We call $D \subseteq [0, T] \times \mathbb{R}^2$ the domain of the state process $X_t$.

The fund manager’s problem is to maximise $V(\Pi)$ over all $\Pi \in \Pi$. We define the optimal value function $V : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ by

$$V(t, X_t) = \sup_{\Pi \in \Pi} V(t, X, \Pi) \quad (2.15)$$

If there exists an admissible control law $\Pi^{opt}$ with the property that $V(\Pi^{opt}) = V$, then we can assert that $\Pi^{opt}$ is the optimal control law for our problem.

3 HJB equation and main results

3.1 HJB equation

We start by deriving the associated Hamilton-Jacobi-Bellman (hence HJB) equation for the optimal value function $V$.

**Proposition 3.1.** Under the following hypothesis:

- there exists an optimal control law $\Pi^{opt}$
- the optimal value function $V(t, S_t, \Psi_t)$ is regular in the sense that $V \in C^{1,2,2}$

the optimal value function $V(t, S_t, \Psi_t)$ satisfies the Hamilton-Jacobi-Bellman equation

$$\begin{cases}
V_t + \sup_{\Pi \in \Pi} \left\{ U(t, X_t, \Pi) + \mathcal{L}^{Pi} V \right\} = 0 & \forall (t, X) \in (0, T) \times \mathbb{R}^2 \\
V(T, X) = U(X_T) & \forall X \in \mathbb{R}^2
\end{cases} \quad (3.1)
$$

The supremum in the HJB equation above is achieved by the optimal control law $\Pi^{opt}(t, X)$

The partial differential equation operator $\mathcal{L}^{\pi} V(t, X_t)$ is the following
\[ L^\pi \mathcal{V}(S_t, \Psi_t, t) = \left( (m^\top \cdot \Pi_t - \mu_B(t) + \sigma_B(t)^2)S_t + \frac{1}{\alpha} (\Psi_t - 1) \right) V_S \\
+ \left( (\mu_T(t) + \mu_B(t) + \sigma_B(t)^2 - \sigma_T(t) \sigma_B(t)) \Psi_t \right) V_\Psi \\
+ \frac{1}{2} \left( S_t^2 (\Pi_t^\top \cdot \Sigma \cdot \Pi_t + \sigma_B(t)^2) - \sigma \cdot \Pi_t \sigma_B(t) S_t^2 \right) V_\Sigma \\
+ \frac{1}{2} \left( (\sigma_T(t) - \sigma_B(t))^2 \Psi_t^2 \right) V_\Psi \\
+ \left( (\sigma_T(t) - \sigma_B(t)) S_t \Psi_t \right) V_S \\
+ \frac{1}{2} \left( S_t^2 \Pi_t^\top \cdot \Sigma \cdot \Pi_t + \sigma_B(t)^2 \right) V_\Sigma \\
+ \frac{1}{2} \left( \sigma_T(t) - \sigma_B(t) \right) S_t \Psi_t V_\Psi \\
+ \frac{1}{2} \left( \sigma_T(t) - \sigma_B(t) \right) S_t \Psi_t V_\Psi \tag{3.2} \]

where \( V_S \) and \( V_\Sigma \) correspond respectively to the first and the second order derivative of the value function with respect to \( S \), while \( V_\Psi \) and \( V_\Psi \Psi \) are partial derivatives with respect to \( \Psi \) and finally \( V_t \) represents the partial derivative with respect to time.

Following Hainaut and Devolder (2006) [8], we can get rid of the constraint (2.4) by expressing the vector \( \Pi_t \) as a matrix product, which turns out to be a function of the first \((n-1)\) elements of \( \Pi_t \):

\[ \Pi_t = M_1 \cdot \pi_t + M_2 \tag{3.3} \]

where

\[
M_1 = \begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
-1 & \cdots & -1
\end{pmatrix} \\
M_2 = \begin{pmatrix}
0 \\
\vdots \\
0 \\
1
\end{pmatrix}
\]

\( M_1 \) is a constant \((n,n-1)\) matrix and \( M_2 \) is a \((n)\) vector. The vector \( \pi_t \) is the vector of the \((n-1)\) first elements of \( \Pi_t \).

### 3.2 The optimal investment strategy

Let us differentiate the Hamiltonian \( \{U(t, X_t, \Pi) + L^\Pi \mathcal{V}\} \) with respect to \( \pi_t \), for a fixed arbitrary point \((t, x) \in [0, T] \times \mathbb{R}^2\) and let it be equal to zero. We thus obtain the optimal investment strategy, denoted by \( \pi_t^{opt} \):

\[
M_1^\top \cdot mS_t V_S + S_t^2 V_\Sigma M_1^\top \cdot \Sigma \cdot (M_1 \cdot \pi_t + M_2) = 0 \tag{3.4}
\]

To simplify the notations, we define \( N_1 \), a \((n-1, n-1)\) matrix, and \( N_2 \), a \((n-1)\) vector

\[
N_1 = M_1^\top \cdot \Sigma \cdot M_1 \\
N_2 = M_1^\top \cdot \Sigma \cdot M_2
\]

From equation (3.4) we find \( \pi_t^{opt} \):

\[
\pi_t^{opt} = N_1^{-1} \cdot \left( - \frac{M_1^\top \cdot mV_S}{S_t V_\Sigma} - N_2 \right) \tag{3.5}
\]
The complete optimal strategy of investment, $\Pi_{t}^{\text{opt}}$, is therefore

$$\Pi_{t}^{\text{opt}} = - \frac{V_{S}}{S_{t}V_{SS}} M_{1} \cdot N_{1}^{-1} \cdot M_{1}^{\top} \cdot m - M_{1} \cdot N_{1}^{-1} \cdot N_{2} + M_{2}$$

To further simplify the notation we define

$$P_{1} = M_{1} \cdot N_{1}^{-1} \cdot M_{1}^{\top}$$
$$P_{2} = M_{1} \cdot N_{1}^{-1} \cdot N_{2} + M_{2}$$

Thus, the complete optimal investment strategy is

$$\Pi_{t}^{\text{opt}} = - \frac{V_{S}}{S_{t}V_{SS}} P_{1} \cdot m - P_{2} \tag{3.6}$$

The same result was obtained by Hainaut and Devolder (2006) [8]. Obviously, the optimal control function depends not only on the choice of $(t, x)$, but also on the function $V$. So, we substitute the equation (3.6) into the PDE (3.1)

$$\begin{cases} 0 = V_{t} + U(t, X_{t}) + [(m^{\top} \cdot (-\frac{V_{S}}{S_{t}V_{SS}} P_{1} \cdot m - P_{2}) - \mu_{B}(t) + \sigma_{B}(t)^{2}S_{t}^{2}((\frac{V_{S}}{S_{t}V_{SS}} P_{1} \cdot m - P_{2})^{\top} \cdot \Sigma \cdot (-\frac{V_{S}}{S_{t}V_{SS}} P_{1} \cdot m - P_{2})) + \sigma_{B}(t)^{2} - \frac{\Sigma}{S_{t}V_{SS}} P_{1} \cdot m - P_{2})^{\top} \cdot \Sigma \cdot \sigma_{B}(t)S_{t}^{2}]V_{SS} \\ V(T, X) = U(X_{T}) \end{cases} \tag{3.7}$$

As already said, we are assuming that the fund manager wishes to continuously maximise the utility arising from the sustainability ratio, by using an adapted investment policy $\Pi_{t}$, which is the control variable of our model. The manager’s preferences are reflected by a quadratic utility function

$$U(S_{t}) = k S_{t} - (1 - k)(S_{t} - TS(t))^{2} \tag{3.8}$$

where $k \in [0, 1]$ is a parameter expressing the fund manager’s preference for a large sustainability ratio and $TS(t)$ is a deterministic function expressing a path for an ideal target sustainability ratio. According to the chosen utility function, the weight $(1 - k)$ rewards ratios which dynamically stay close to the target ratio.

As the utility function is quadratic, we postulate that the solution $V(t, S_{t}, \Psi_{t})$ shares the same quadratic structure

$$V(t, S_{t}, \Psi_{t}) = a(t, \Psi)S_{t}^{2} + b(t, \Psi)S_{t} + c(t, \Psi) \tag{3.9}$$

Then

$$V_{S} = 2a(t, \Psi)S_{t} + b(t, \Psi) \tag{3.10}$$
$$V_{SS} = 2a(t, \Psi) \tag{3.11}$$
\[ V_t = \partial_t a(t, \Psi) S_2^2 + \partial_t b(t, \Psi) S_1 + \partial_t c(t, \Psi) \] (3.12)

Let us insert these derivatives into (3.7). We thus deduce the following three partial differential equations for \( a(t, \Psi) \), \( b(t, \Psi) \) and \( c(t, \Psi) \):

\[
\begin{align*}
\partial_t a(t, \Psi) &= -2a(t, \Psi) \left( -\mu_B(t) - m^\top P_2 + P_2^\top \Sigma \cdot P_1 \cdot m - m^\top (P_1 \cdot m) \right) \\
&- a(t, \Psi) \cdot \left[ P_2^\top \Sigma \cdot P_2 + \sigma_B(t)^2 + m^\top (P_1 \cdot m) - \frac{1}{2} \sigma \sigma_B(t) \cdot (P_1 \cdot m) \right] \\
&+ \left( P_1 \cdot m \right)^\top + P_2^\top \right] + (1 - k) (3.13) \\
\partial_t b(t, \Psi) &= -b(t, \Psi) \left[ -\mu_B(t) - m^\top P_2 + \sigma_B(t)^2 + P_2^\top \Sigma \cdot P_1 \cdot m \right. \\
&+ (m^\top P_1^\top \Sigma \cdot P_1 \cdot m) - 2m^\top (P_1 \cdot m) - \frac{1}{2} \sigma \sigma_B(t) \cdot (P_1 \cdot m) \left. \right] \\
&- 2a(t, \Psi) \left( \frac{1}{\alpha} (\Psi - 1) \right) - k - 2(1 - k) TS(t) (3.14) \\
\partial_t c(t, \Psi) &= -b(t, \Psi) \left[ \frac{1}{\alpha} (\Psi - 1) \right] + \frac{b(t, \Psi)^2}{2a(t, \Psi)} \cdot m^\top m \\
&- \frac{1}{2} (P_1 \cdot m)^\top \Sigma \cdot P_1 m + (1 - k) TS(t)^2 (3.15)
\end{align*}
\]

with the terminal conditions

\[
\begin{align*}
a(T, \Psi) &= -(1 - k) (3.16) \\
b(T, \Psi) &= k + 2(1 - k) TS(T) (3.17) \\
c(T, \Psi) &= -(1 - k) TS(T)^2 (3.18)
\end{align*}
\]

and the boundary conditions

\[
\begin{align*}
a(t, \Psi_1) &= -(1 - k) (3.19) \\
b(t, \Psi_1) &= k + 2(1 - k) TS(t) (3.20) \\
c(t, \Psi_1) &= -(1 - k) TS(t)^2 (3.21) \\
a(t, \Psi_2) &= -(1 - k) (3.22) \\
b(t, \Psi_2) &= k + 2(1 - k) TS(t) (3.23)
\end{align*}
\]
where $\Psi_1$ and $\Psi_2$ are the bounds of the function $\Psi_t$. Solving them via a numerical method (Euler-Cauchy method, see Cushing (2004) [7]), we find the optimal investment strategy

$$\tilde{\Pi}^{opt}_{t} = -\frac{\bar{V}_S}{S_t V_{SS}} P_1 \cdot m - P_2$$

where the symbol $\tilde{}$ denotes the result of the numerical integration.

### 3.3 The verification theorem

**Proposition 3.2.** Given the optimal portfolio $\Pi^{opt}_{t} = M_1 \cdot \pi^{opt}_{t} + M_2$, the value function $V(t, S_t, \Psi_t)$ is

$$V(t, S_t, \Psi_t) = a(t, \Psi) S^2_t + b(t, \Psi) S_t + c(t, \Psi)$$

where functions $a(t, \Psi)$, $b(t, \Psi)$, $c(t, \Psi)$ are solutions of the differential equations

$$a'(t, \Psi) = -2a(t, \Psi) (-\mu_B(t) - m^\top \cdot P_2 + \sigma_B(t)^2 - m^\top \cdot P_1 \cdot m)$$

$$- a(t, \Psi)[\sigma_B(t)^2 + (P_1 \cdot m)^\top \cdot \Sigma \cdot P_1 \cdot m + (P_1 \cdot m)^\top \cdot \Sigma \cdot P_2 + P_2^\top \cdot \Sigma \cdot P_1 \cdot m + P_2^\top \cdot \Sigma \cdot P_2] - \sigma_B(t) a(t, \Psi) ([P_1 \cdot m)^\top + P_2^\top) + (1-k)$$

$$b'(t, \Psi) = -b(t, \Psi) (-m^\top \cdot P_1 \cdot m - m^\top \cdot P_2 - \mu_B(t) + \sigma_B(t)^2)$$

$$- 2a(t, \Psi) \left( -m^\top \cdot g(t, \Psi) + \frac{1}{\alpha}(\Psi_t - 1) \right)$$

$$- a(t, \Psi) [ (P_1 \cdot m)^\top \cdot \Sigma \cdot g(t, \Psi) + P_2^\top \cdot \Sigma \cdot g(t, \Psi) + g(t, \Psi)^\top \cdot \Sigma \cdot P_1 \cdot m + g(t, \Psi)^\top \cdot \Sigma \cdot P_2] - \sigma \cdot \sigma_B(t) a(t, \Psi) g(t, \Psi)^\top - k - 2(1-k) \cdot TS(t)$$

$$c'(t, \Psi) = -b(t, \Psi) \left( -m^\top \cdot g(t, \Psi) + \frac{1}{\alpha}(\Psi_t - 1) \right)$$

$$- a(t, \Psi) g(t, \Psi)^\top \cdot \Sigma \cdot g(t, \Psi) + (1-k)TS(t)^2$$

and $g(t, \Psi_t)$ is defined as:

$$g(t, \Psi_t) = -S_t(\Pi^{opt}_{t, \Psi} + P_1 \cdot m + P_2)$$

with the terminal conditions (3.16), (3.17), (3.18) and the boundary conditions (3.19), (3.20), (3.21), (3.22), (3.23), (3.24).
Proof. Given the optimal asset allocation \( \Pi^{opt}_t = M_1 \cdot \pi_1^{opt} + M_2 \), by defining \( g(t, \Psi_t) \), we find:

\[
\Pi(t, \Psi_t) = -\frac{1}{S_t}(S_tP_1 \cdot m + g(t, \Psi_t)) \tag{3.30}
\]

Moreover, if we assume that the value function \( V(t, S_t, \Psi_t) \) is a quadratic one like (3.9), the Bellman’s equation becomes

\[
0 = \partial_t a(t, \Psi)S_t^2 + \partial_t b(t, \Psi)S_t + \partial_t c(t, \Psi) + U(t, X_t) + \\
((m^\top \cdot (-\frac{1}{S_t}(S_tP_1 \cdot m + g(t, \Psi_t)) - P_2) - \mu_B(t) \\
+ \sigma_B(t)^2)S_t + \frac{1}{\alpha}(\Psi_t - 1))(2a(t, \Psi)S_t + b(t, \Psi)) + \\
\frac{1}{2}S_t^2 \cdot 2a(t, \Psi) \cdot ((-\frac{1}{S_t}(S_tP_1 \cdot m + g(t, \Psi_t)) - P_2)^\top. \\
\Sigma \cdot (-\frac{1}{S_t}(S_tP_1 \cdot m + g(t, \Psi_t)) - P_2) + \sigma_B(t)^2) \\
- \frac{1}{2} \cdot \sigma(-\frac{1}{S_t}(S_tP_1 \cdot m + g(t, \Psi_t)) - P_2)^\top \sigma_B(t)S_t^2 \cdot 2a(t, \Psi)
\]

Regrouping the terms in \( S^2 \), \( S \) and independent of \( S \) gives us the system of equations (3.26), (3.27), (3.28) that is equivalent to the system (3.13), (3.14), (3.15).

\[
4 \text{ Illustrative example}
\]

To illustrate the model developed in the previous section, we will look at data from a real Italian Professional Order pension fund. Our model parameters are related both to pension fund cash flows and to financial market data.

Subsections 4.1 and 4.2 below show the inputs we used in this example, while subsection 4.3 sets out our main results.

4.1 Pension Fund data

Table 1 summarises the current state of the pension fund under consideration. As we can see, at time \( t = 0 \) this fund is in a position of actuarial surplus in the sense that the level of contributions exceeds the level of benefits (see the value of \( \Psi \)). Of course, this situation may change in the future depending on stochastic contributions and benefit paths. Indeed, the sustainability ratio \( S \) is less than one, i.e. the pension fund is not in sustainability equilibrium. The current benefits multiplier \( \alpha \) is set at 5, according to the ratio monitored by the Italian Ministry.

We used polynomial estimations for functions representing the average benefits \( \mu_B(t) \), average contributions \( \mu_C(t) \) and the target sustainability ratio path \( TS(t) \). Estimations are based on actuarial calculations of the pension fund’s...
Table 1: Data inherent an Italian Professional Order pension fund (year 2009)

<table>
<thead>
<tr>
<th>Pension fund's wealth (A)</th>
<th>1,600.00 mill.€</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contributions (Γ)</td>
<td>423.90 mill.€</td>
</tr>
<tr>
<td>Benefits (B)</td>
<td>365.48 mill.€</td>
</tr>
<tr>
<td>Sustainability ratio (S) (α = 5)</td>
<td>0.88</td>
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<tr>
<td>ψ</td>
<td>1.16</td>
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</tbody>
</table>

Table 2: Parameters of polynomial estimates for average benefits, contributions and target sustainability ratio

<table>
<thead>
<tr>
<th></th>
<th>( \mu_B(t) )</th>
<th>( \mu_{Γ}(t) )</th>
<th>( TS(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( β_0 )</td>
<td>( 3.9340 \cdot 10^{-4} )</td>
<td>( 4.0085 \cdot 10^{-4} )</td>
<td>( 7.7883 \cdot 10^{-4} )</td>
</tr>
<tr>
<td>( β_1 )</td>
<td>( -1.2830 \cdot 10^{-2} )</td>
<td>( 1.9198 \cdot 10^{-2} )</td>
<td>( 6.4384 \cdot 10^{-2} )</td>
</tr>
<tr>
<td>( β_2 )</td>
<td>( 5.0009 \cdot 10^{-4} )</td>
<td>( -3.5416 \cdot 10^{-4} )</td>
<td>( -4.5144 \cdot 10^{-4} )</td>
</tr>
<tr>
<td>( β_3 )</td>
<td>( -2.7166 \cdot 10^{-4} )</td>
<td>( -3.0560 \cdot 10^{-5} )</td>
<td>( -1.0694 \cdot 10^{-5} )</td>
</tr>
<tr>
<td>( β_4 )</td>
<td>( 8.4160 \cdot 10^{-6} )</td>
<td>( 3.1394 \cdot 10^{-6} )</td>
<td>( 4.4130 \cdot 10^{-6} )</td>
</tr>
<tr>
<td>( β_5 )</td>
<td>( -1.4149 \cdot 10^{-7} )</td>
<td>( -6.7437 \cdot 10^{-8} )</td>
<td>( -8.4781 \cdot 10^{-8} )</td>
</tr>
<tr>
<td>( β_6 )</td>
<td>( 9.8623 \cdot 10^{-10} )</td>
<td>( 4.5602 \cdot 10^{-10} )</td>
<td>( 4.9922 \cdot 10^{-10} )</td>
</tr>
</tbody>
</table>

future cash flows (the actuarial balance sheet). Under the Italian legislation on Professional Order pension funds, actuarial balances must be drawn up to monitor the evolution of the fund in the long run (up to 50 years). Table 2 shows the parameters of the polynomial regressions for the above-mentioned functions. Subscripts of betas refers to the degree of the variable \( t \) to which they are related. Figures 1 and 2 show respectively the expected values for future cash flows and for the sustainability ratio.

The volatility of benefit rate \( σ_B(t) \) and of contributions rate \( σ_{Γ}(t) \) are set to 50% of respectively \( \mu_B(t) \) and \( \mu_{Γ}(t) \).

4.2 Financial market data

We assume that the asset manager aims to invest in 15 financial assets whose expected returns, volatilities and correlations are described respectively in Tables (3), (4) and (5).
### Table 3: Expected values of asset classes

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Expected Value</th>
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</thead>
<tbody>
<tr>
<td>Monetary</td>
<td>3.5%</td>
</tr>
<tr>
<td>Bond Gov EMU</td>
<td>4.55%</td>
</tr>
<tr>
<td>Bond Gov World ex EMU</td>
<td>4.55%</td>
</tr>
<tr>
<td>Bond Corp. EU</td>
<td>5.16%</td>
</tr>
<tr>
<td>Bond Corp. USA</td>
<td>5.34%</td>
</tr>
<tr>
<td>Bond Corp. High Yield</td>
<td>6.30%</td>
</tr>
<tr>
<td>Bond Emerg. Markets</td>
<td>6.83%</td>
</tr>
<tr>
<td>Inflation</td>
<td>4.64%</td>
</tr>
<tr>
<td>Equity EU</td>
<td>8.40%</td>
</tr>
<tr>
<td>Equity USA</td>
<td>8.93%</td>
</tr>
<tr>
<td>Equity Pacific</td>
<td>7.96%</td>
</tr>
<tr>
<td>Equity Emerging markets</td>
<td>15.15%</td>
</tr>
<tr>
<td>Hedge Funds</td>
<td>6.65%</td>
</tr>
<tr>
<td>Commodities</td>
<td>9.27%</td>
</tr>
<tr>
<td>Private Equity</td>
<td>12.16%</td>
</tr>
</tbody>
</table>

### Table 4: Volatility of asset classes

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monetary</td>
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</tr>
<tr>
<td>Bond Gov EMU</td>
<td>5.0%</td>
</tr>
<tr>
<td>Bond Gov World ex EMU</td>
<td>4.9%</td>
</tr>
<tr>
<td>Bond Corp. EU</td>
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<tr>
<td>Bond Corp. USA</td>
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<tr>
<td>Bond Corp. High Yield</td>
<td>7.7%</td>
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<tr>
<td>Bond Emerg. Markets</td>
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<tr>
<td>Inflation</td>
<td>2.5%</td>
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<tr>
<td>Equity EU</td>
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<td>Equity USA</td>
<td>19.4%</td>
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<td>Equity Pacific</td>
<td>20.8%</td>
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<tr>
<td>Equity Emerging markets</td>
<td>23.0%</td>
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<tr>
<td>Hedge Funds</td>
<td>8.2%</td>
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<tr>
<td>Commodities</td>
<td>19.5%</td>
</tr>
<tr>
<td>Private Equity</td>
<td>24.2%</td>
</tr>
</tbody>
</table>
Figure 1: Actuarial Estimations on Contribution and Benefits Income

Figure 2: Actuarial Estimations on Target Sustainability ratio
Table 5: Correlation matrix of asset classes

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</table>

16
4.3 Results

We assume that the asset manager sets his investment horizon at fifteen years \( (T = 15) \); this is a long term time span according to pension funds’ characteristics. In order to correctly describe the asset manager’s risk aversion, we set the risk aversion coefficient \( k \) as equal to two different values, namely to 0.1 and to 0.6. The first condition is related to a more risk-adverse investor. Figure 3 presents the optimal proportions of the fund, invested in the 15 asset classes detected in the previous subsection, for the coefficient \( k = 0.1 \), while Figure 4 presents the value related to the asset class Equity EU.

Figure 3: Asset mix, \( k = 0.1, \Psi = 1 \)

Figure 4: Quota invested in Equity EU, \( k = 0.1, \Psi = 1 \)

Figures (5) and (6) show the same results in the case \( k = 0.6 \).

5 Final remarks

In this paper, we have investigated the optimal dynamic portfolio allocation problem for a pension fund which operates in a mixed funded/PAYG financing regime. This is the case for a particular set of first-pillar Italian private
closed pension schemes. To take into account the sustainability of the fund, as recommended by the Italian regulator, we derived the stochastic dynamic for an empirical indicator which is the ratio of the fund wealth to a multiple of the current expenditure for pensions. This sustainability ratio $S_t$ acts as the state variable in a stochastic optimal control problem. Indeed, our model has two state variable, crucially depending also on the balance between active and retired members of the plan $\Psi_t$.

On the one hand, we assumed that the asset manager aims to maximise this empirical ratio, but on the other hand we supposed he has an incentive to stay close to a certain target sustainability ratio path. We thus obtained closed-form expressions for the investment strategy depending on the value function.

Finally, to determine the optimal asset allocation we solve a system of three PDEs by a numerical approximation method.
Acknowledgements

The authors would like to thank Prof. Tiziano Vargiolu, of the University of Padova, Italy, for valuable comments.

References


