Empirically Effective Bond Pricing Model for USGBs and Analysis on Term Structures of Implied Interest Rates in Financial Crisis

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Abstract

Using Kariya’s bond pricing (1993) model, this paper makes a comprehensive empirical analysis on US Government bond (USGB) prices for a period including the Financial Crisis in 2008. The model is a cross-sectional model that simultaneously values individual fixed-coupon (non-defaultable) bonds of different coupon rates and maturities via a stochastic discount function approach. First we briefly clarify the theoretical relation between our stochastic discount function approach and the interest rate (spot rate or forward rate) approach in mathematical finance. Then we make a comprehensive empirical study on its pricing capability for individual USGBs with different attributes and on its capacity of describing the movements of term structures of interest rates that USGBs imply as yield curves. Based on various tests of validity in GLS (Generalized Least Squares) framework we propose a specific formulation with a polynomial of order 6 for the mean discount function that depends on maturity and coupon as attributes and a specific covariance structure. It is shown that even in the middle of the Financial Crisis, the cross-sectional model we propose is shown to be very effective for simultaneously pricing all the existing USGBs and deriving and describing zero yields.

Key words and phrases: Cross-Sectional Bond Pricing Model, Term Structure of Interest Rates, Subprime Shock, Financial Crisis, Swap Rate, US Government Bond, Generalized Least Squares, Forward Rate, Discount Function

1 Introduction

Using the Kariya (1993) bond pricing (KBP hereafter) model with stochastic discount function, Kariya and Tsuda (1994,96) made an empirical analysis on Japanese Government bond (JGB) with a limited setting where the term structure of mean discount function was approximated by a polynomial of order 2 and sample period was from 1980.1 to 1992.12. Recently Kariya, Wang, Wang, Doi and Yamamura (KWWDY)
(2012) made a fully comprehensive study on JGBs with a general setting for sample period 2005.9 through 2010.8 including the Financial Crisis and found out that the cross-sectional market model that takes bond attributes into account effectively values JGB prices. In this paper, with the same viewpoint we make a comprehensive empirical analysis on the US Government bond (USGB) prices and on term structures of implied interest rates for the period from 2006.4 through 2011.3 including (60 months) including the Financial Crisis Period 2008-2009. The term structure movement implied by the model is compared to that of the swap rates.

A main feature of this model is that a stochastic realization of each individual GB price at present time 0 is viewed as equivalent to a realization at 0 of the whole stochastic process of the random cash-flow discount rate, which is defined on its term to maturity and depends on attributes of each individual bond such as coupon rate, term to maturity, etc. Another feature associated with this feature is that the cross-sectional correlation structure of all the GB prices at 0 is obtained through that of the corresponding random discount functions. Though this model is a cross-sectional model as it stands, it can be extended to certain types of dynamic models with time varying coefficients or equivalently state-space models, which will be discussed elsewhere.

In our paper the arbitrage-free paradigm with time-continuous approach in mathematical finance is not adopted because the paradigm inevitably requires that the model is diffusion (Markovian) model. In fact, actual interest rate processes in practice move with business cycles, which are not Markovian. This excludes spot interest model such as CIR (Cox-Ingersoll-Ross) model etc., which will be unable to value all the existing GBs at 0 of different maturities and coupons simultaneously with the correlations being taken into account. This is because it cannot describe differences of bond prices due to such attributes as coupon rate and maturity. On the other hand, in modeling a swap rate process Collin-Dufrense and Solnik (2001) and Feldhutter and Lando (2007) take the dependence of swap rates on credit attributes into account and specify a swap rate process as the sum of an abstract risk-free rate process and a convenience yield process where they are assumed to be independent. Here the convenience yield is supposed to represent such attributes as liquidity premium, credit premium (collateral condition), etc. Recent developments in the area of interest rate models in mathematical finance are found in Brigo and Mercurio (2006) and Filipovic (2009). While, in Anderson, et al (1996) the derivation and estimation of yield curves in a traditional or practical approach is well exposited from a recent perspective.

For its important implication, let us discuss on the empirical observation that bond prices formed in the market are affected by the difference of some attributes such as
maturity and coupon. This observation will be significantly confirmed even for USGB prices as will be shown in this paper. This will imply that main objectives of holding government bonds of different attributes in investors’ portfolios are not those of making arbitrage activities as the math finance theory claims, but investors’ own objectives such as asset-liability management as in life insurance or pension fund. In fact, a specific cash flow pattern guaranteed by holding certain attribute-specific GBs will be a big value to such institutional investors because they need to match cash inflow with cash outflow over a long time horizon. For them, coupon and maturity are important attributes that affect investment decisions over a future perspective. Hence for example, depending on cash inflow-outflow structures of investors and on future perspectives on movements of interest rates, it may happen that a GB of 2 year maturity and 5% coupon is less preferred to a GB of 6 year maturity and 3% coupon, implying that coupon and maturity are interdependent attributes. It is noted that those institutional investors do not necessarily prefer de-coupon or stripped bonds engineered by investment banking as making a portfolio from these stripped bonds to match the cash inflows and outflows is costly and involves additional credit risk of investment bankers. In short, hypotheses of no maturity effect and/or no coupon effect are strongly rejected even for such supposedly efficient US bond markets.

In KWWDY (2012) the capacity of the CSM model was comprehensively studied in view of its pricing capability for individual JGBs with different attributes and in view of its capacity of describing the movements of term structures of interest rates that JGBs imply as yield curves, where the sample period is from 2005.9 till 2010.8 (60 months) including the financial crisis period, where the parameters are estimated by GLS (Generalized Least Squares) method. They proposed a formulation with a polynomial of order 6 approximating the mean discount function for all times and with a specific covariance structure describing the cross-sectional correlations of JGB prices.

In this paper we also make a similar comprehensive empirical study on the USGB prices and make an extensive comparison. Our sample period includes the Financial Crisis Period starting from 2008.8 and each monthly set of data contains about 140 bond prices. For each monthly set of data, we consider 4 models, which we call M0, M1, M2 and M3. Here M0 is a model that is attribute-independent and use it as a base model to derive the term structure of interest rates as well as to discuss the relative performances of the other attribute-dependent models: M1 with maturity effect, M2 with coupon effect and M3 with maturity and coupon. Even in the Financial Crisis M3 is shown to be of great capability in pricing all USGBs. The term structures of interest rates that M0 implies are shown to perform well by comparing them with swap rates
and hence they can be used for various purposes in practice.

In Section 2 we briefly describe a theoretical framework for the stochastic discount function approach to GB pricing and clarify the relationship between the interest rate model approach in mathematical finance and our discount function model approach to GB pricing, which are both attribute-dependent. This will help us freely use an unconditional non-Markovian model of stochastic discount function without specifying interest rate model and makes it possible to derive a term structure of interest rates implied by the GBs.

In Section 3 our GB model is specified in detail for empirical work and an estimation procedure using Generalized Least Squares method is proposed as in Kariya (1993) (see also Kariya and Kurata (2004)). The important parts are the specifications of the mean discount function and the covariance structure of bond prices, which separates our model from the other models. The mean discount function is uniformly approximated by a polynomial with attribute variables in its coefficients. Of course, there are alternative specifications for the mean discount function such as spline-type or exponential-type specification which includes Nelson-Siegel (1987). But we stick to a simple polynomial specification for robustness and stability of our linear-parametric specification in prices as we aim both to price individual GBs and to derive the term structure of interest rates. This model can be used for GB trading purpose as well as for risk management of debts.

In Section 4 we set up our empirical framework. Each set of monthly cross-sectional data in the 5 years from 2006.4 through 2011.3 consists of about 144 USGB prices. To compare our cross-sectional results in time series, we divide our sample period into 4 periods: upturn economy, downturn economy, Financial Crisis and Post Crisis. Trading volumes and term structure of coupons are observed.

In Section 5, based on M0, the polynomial for the mean discount function is empirically found to be of order 6 overall for 60 cross-sectional models. The criteria for model selection are (1) the AIC, (2) the minimized GLS value and (3) the residual standard deviation (RSD). The selected order is used for all the models so that M0, M1 and M2 are a sub-model of M3. In Section 6, based on (2) and (3), the performances of our models are compared to find that M3 is the best in pricing USGBs, but M0 is also good enough to price USGBs. In fact, even in the worst case of 2008.12 in the middle of the Financial Crisis, the S-RSDs of M0, M1, M2 and M3 are 1.2085, 0.8448, 0.8739 and 0.7337 (dollar).

In Section 7, using $F$-ratios based on the minimized value of the GLS objective function, hypotheses of no maturity effect and/or no coupon effect are tested. In fact, hypotheses
of M0 vs M1, M0 vs M2 and M1 vs M3 are shown to be both strongly significant, confirming that M3 is the best in pricing USGBs. Interestingly the significance in the F-ratios is much stronger than the case of JGBs, implying that investors’ selections of bonds are more keen about the differences of the bond attributes, which is an evidence against the arbitrage-free paradigm.

In Section 8, we investigate on the changes of cross-sectionally estimated parameters in the discount function of M0 and in the covariance matrix change over time and on the changes of the correlation structure over time. The standard deviations and correlations of individual prices are computed and it is shown that the larger the maturity or coupon is, the larger the standard deviation and that the correlations are getting larger in the Financial Crisis Period.

In Section 9, time series variations of the cross-sectional term structures of about 140 residuals over 60 months are discussed and some influences of the Financial Crisis on the performances of our models are observed.

In Section 10 we consider the effectiveness of our models in terms of the term structures of implied interest curves obtained via the term structures of discount curves, which are given by M0, M1, M2 and M3. First in the worst time of 2008.12 the term structures of interest rates that M1, M2 and M3 models imply are defined and compared to that of M0. Those of M1 and M2 are influenced by maturity effect and coupon effect and are slightly smaller than the term structures of M0, though those of M3 may not be appropriate. Second, it is observed that the M0 term structures of interest rates perform well for all times. Then they are compared to those of dollar-dollar swap rates and the credit quality of USGBs relative to that of swap rates, which is defined as the spreads of swap rate minus bond rare, changes over 60 months and the spreads of longer maturities are shown to get negative in the Financial Crisis and thereafter.

2 Pricing Non-defaultable Bonds

Suppose that at time $t$ there are $G$ non-defaultable bonds whose face values are all 100 dollars and whose prices at $t$ are denoted by $P_g$ ( $g = 1, \cdots, G$). Here by letting $t = 0$ denote the time of each cross-sectional analysis, we omit $t$ from the suffixes of variables unless otherwise specified, and so $G = G_0, P_g = P_{g0}$ and so on. Viewed from $t = 0$, let

$$s_{g_1} < s_{g_2} < \cdots < s_{gM(g)} \quad (g = 1, \cdots, G)$$

(2.1)
denote the future time points at which the \( g \)-th bond generates the cash flows (CFs) (coupons or principal). Here \( s_{gj} \)'s are measured in years, where \( s_{gM(g)} \) is the term to maturity (or simply maturity) of the \( g \)-th bond. Let \( c_g \) be its coupon rate (dollars). If coupons are paid biannually, the CF function \( C_g(s) \) of the \( g \)-th bond is expressed as
\[
C_g(s) = \begin{cases} 
0.5c_g & (s = s_{gm}, m \neq M(g)) \\
100 + 0.5c_g & (s = s_{gM(g)}) \\
0 & (s \neq s_{gm})
\end{cases}.
\]
(2.2)

However, in our argument the cash flow function can be arbitrary, so long as the future CFs and their CF times are given in advance.

Let \( D_g(s) \) be the attribute-dependent stochastic discount function of the \( g \)-th bond defined on common region \( 0 < s \leq s_{aM(a)} \) with \( s_{aM(a)} = \max_g s_{gM(g)} \), where the whole values of \( D_g(s) \) are realized all at \( t = 0 \) and \( D_g(s) \) discounts CF \( C_g(s) \) by \( C_g(s)D_g(s) \). Note \( C_g(s) = 0 \) if \( s > s_{gM(g)} \). Under these notations, our basic formulation for modeling \( G \) bond prices simultaneously at \( t = 0 \) is based on the following expression:
\[
P_g = \sum_{j=1}^{M(g)} C_g(s_{gj})D_g(s_{gj}) \quad (g = 1, \ldots, G).
\]
(2.3)

This is an unconditional cross-sectional model for prices. In (2.3), we regard the realization of each price \( P_g \) as equivalent to the realization of the whole function \( \{D_g(s) : 0 \leq s \leq s_{aM(a)}\} \) where \( g = 1, \ldots, G \). In fact, the realizations of \( G \) bond prices correspond to those of the rates \( \{D_g(s_{gj}) : j = 1, 2, \ldots, M(g), \ g = 1, 2, \ldots, G\} \) and the correlation structure of these stochastic discount rates implies those of prices.

In the spot rate approach in mathematical finance a non-defaultable bond price at \( t = 0 \) is specified as the conditional expectation, which is expressed as follows: for each individual price
\[
P_g(1) = \sum_{j=1}^{M(g)} C_g(s_{gj})\overline{D}(s_{gj}) \quad \text{with}
\overline{D}(s_{gj}) = E_0[\exp(-\int_0^1 r_u du)] = H(r_{l_g}, s_{gj}, \theta),
\]
(2.4)

where \( \{r_u : 0 \leq u \leq s_{aM(a)}\} \) is a process of instantaneous spot interest rates \( \{r_u\} \) that is
common to all the bonds and $E_0[\cdot]$ denotes the conditional expectation given $r_0$ and its past values at 0 with respect to a risk neutral measure. But the measure is not uniquely identified. Here $\theta$ denotes a set of possible parameters when a specific model such as CIR model or Vasicek model is used for the spot rate process $\{r_t\}$. In mathematical finance the conditional expectation given $r_0$ can be regarded as being taken under a risk neutral measure and (2.4) is claimed to hold a.s. for all the bonds in its no-arbitrary theory. However in reality it does not follow that (2.4) holds a.s. for all bonds given $r_0$. In such a spot rate approach, modeling the spot rate process yields the conditional discount function in (2.4), and the zero yield curve $\{R_u: 0 \leq u \leq s_{M(\omega)}\}$ defined by

$$H(r_0, s, \theta) = \exp(-R_s) \text{ or equivalently } R_s = -\frac{1}{s} \log H(r_0, s, \theta).$$

In the sequel we use the real measure that generates real data.

Empirically speaking, it is often observed that bond prices formed in the market depend on such attributes as coupon and maturity, in which case it is required to take into account such attribute-dependency in the specification of the spot rate process. In modeling a swap rate process Collin-Dufrense and Solnik (2001) and Feldhutter and Lando (2007) take the dependence of swap rates on credit attributes into account and specify a swap rate process as the sum of an abstract risk-free rate process $\{x_t\}$ and a convenience yield process $\{x_{gs}\}; r_{gs} = x_{is} + x_{gs}$, where they are assumed to be independent. Here the convenience yield represents such attributes as liquidity premium, credit premium (collateral condition), etc. The attribute-dependent convenience process $\{x_{gs}\}$ can play an adjusting factor for fitting the model as it can be arbitrarily specified. Using this process the discount function in (2.4) becomes attribute-dependent:

$$D_{s}(s_{gs}) = E_0[\exp(-\int_0^{s_{gs}} r_{gs} ds)] = E_0[\exp(-\int_0^{s_{gs}} (x_{is} + x_{gs}) ds)].$$

In this paper we take an unconditional price model approach that may be regarded as equivalent to a forward rate approach with unconditional expression. In fact, making instantaneous forward rate process attribute-dependent model at $t = 0$ as $\{f_{gs}: 0 \leq s \leq s_{M(\omega)}\}$, the model is expressed unconditionally:
(2.5) \[ P_g \left( s \right) = \sum_{j=1}^{M(g)} C_g(s_j) D_g(s_j) \] with \[ D_g \left( s \right) = \exp\left(-\int_0^s f_{g(s)} du \right). \]

Note that \( \left\{ \int_0^s f_{g(s)} du : 0 \leq s \leq s_{aM(\omega)} \right\} \) gives a term structure process of interest rates realized all at \( t = 0 \). In other words, for each \( g \) a realization of \( P_g \) corresponds to that of the whole path \( \left\{ f_{g(s)} : 0 \leq s \leq s_{aM(\omega)} \right\} \). Though our model corresponds to (2.5), we do not make specification on \( \left\{ f_{g(s)} \right\} \), but we take a price approach in such a way that \( D_g \left( s \right) \) is decomposed into the sum of the mean part and the rest.

On the other hand, one may use the time-continuous HJM (Heath-Jarrow-Morton (1992)) model as an attribute-free forward rate process of term structures. The HJM model is specified conditionally and the Markovian expression is usually explored with the no-arbitrage argument. Even in this case, one single path \( \left\{ f_s : 0 \leq s \leq s_{aM(\omega)} \right\} \) at \( t = 0 \) does not make (2.5) hold a.s. for all \( g \) either. Besides, it is not easy to extend the model to an attribute-dependent model. In Kamizono and Kariya (1998), an empirical research was done for Japanese future data with a specification of the covariances depending on forward rates with the no-arbitrage condition where the market price of risk is estimated.

3 GB Pricing Model

In this section, we specify an attribute-dependent GB pricing model with stochastic discount function.

First, let \( D_g \left( s \right) \) be decomposed into the mean and stochastic deviation functions as

\[
(3.1) \quad D_g \left( s \right) = \bar{D}_g \left( s \right) + \Delta_g \left( s \right).
\]

Substituting this into (2.5), it follows that

\[
(3.2) \quad P_g = \sum_{m=1}^{M(g)} C_g(s_{gm}) \bar{D}_g \left( s_{gm} \right) + \eta_g
\]

\[
\eta_g = C_g \left( s_{gm} \right) \Delta_g \left( s_{gm} \right),
\]

where
3.1 Specification of Mean Discount Function.

The expression (3.2) corresponds to the case in (2.5), but here without specifying the attribute-dependent forward rate process \( \{ f_{s_t} : 0 \leq s \leq s_{aM(a)} \} \), the corresponding mean discount function \( D_g(s) \) on \([0, s_{aM(a)}]\) is assumed to be continuous in \( s \) and then it is uniformly approximated by a \( p \)-th order polynomial:

\[
(3.3) \quad D_g(s) = 1 + (\delta_{11} z_{1g} w_1 + \delta_{12} z_{2g} w_2 + \delta_{13} z_{3g} w_3) s + \cdots + (\delta_{p1} z_{1g} w_1 + \delta_{p2} z_{2g} w_2 + \delta_{p3} z_{3g} w_3)s^p,
\]

where \( z_{1g} = 1, \ z_{2g} = s_{aM(g)} \) and \( z_{3g} = c_g \) with \( w_1 = 1, w_2 = 0 \) or \( 1 \), and \( w_3 = 0 \) or \( 1 \) are the attribute variables of the \( g \)-th bond. It is noted that the parameters of each model are common to all the bonds for \( g = 1, \ldots, G \) and hence they are estimable with \( G \) bond prices, so long as \( G \) is greater than the number of the parameters contained.

In this expression we classify the 4 specifications for the attribute-dependent mean discount function with respect to \( (w_1, w_2, w_3) \):

\[
(3.3a) \quad (1) \text{ M0 Model of } (1,0,0): \text{ the base model of no attribute effect,}
(2) \text{ M1 Model of } (1,1,0): \text{ Model 0 + maturity effect,}
(3) \text{ M2 Model of } (1,0,1): \text{ Model 0 + coupon effect, and}
(4) \text{ M3 Model of } (1,1,1): \text{ Model 0 + maturity effect + coupon effect.}
\]

These four Models are empirically investigated with USGB price data in the sequel.

M0 is our base model to derive term structures of interest rates implied by GB prices. In fact, it yields attribute-independent mean discount function \( \bar{D}(s) \) and it is converted to a yield curve by

\[
(3.4) \quad R_s = -\frac{1}{s} \log \bar{D}(s), \quad 0 \leq s \leq s_{aM(a)},
\]

which is often referred to as a risk-free yield curve in practice, though in fact it is not necessarily risk-free.

Now to get a final form for data, substituting (3.3) into (3.2) yields

\[
\sum_{m=1}^{M(g)} C_g(s_{gm}) \bar{D}_g(s_{gm}) = a_g + (\delta_{11} d_{g11} + \delta_{12} d_{g21} + \delta_{13} d_{g31}) + \cdots + (\delta_{p1} d_{g1p} + \delta_{p2} d_{g2p} + \delta_{p3} d_{g3p}),
\]
where
\[
a_g = \sum_{m=1}^{M(g)} C_g(s_{gm}) \quad \text{and} \quad d_{gij} = \sum_{m=1}^{M(g)} C_g(s_{gm}) z_{gj} s^f_{gm}.
\]

Here \( i \) in \( d_{gij} \) denotes the attribute suffix and \( j \) the polynomial order. Thus letting
\[
\tilde{x}_g = (d_{g11}, d_{g12}, d_{g13}, \ldots, d_{g1p}, d_{g21}, d_{g22}, \ldots, d_{g2p}, d_{g31}, \ldots, d_{g3p})' 3p \times 1 \quad \text{and}
\]
\[
X = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_G)' 3 \times 3p,
\]

we have a regression model
\[
y = X \beta + \eta,
\]
where \( y = (y_1, y_2, \ldots, y_G)' 3 \times 1 \) with \( y_g = P_g - a_k \), \( \eta = (\eta_1, \ldots, \eta_G)' \) and
\[
\beta = (\delta_{11}, \delta_{12}, \delta_{13}, \delta_{21}, \delta_{22}, \delta_{23}, \ldots, \delta_{p1}, \delta_{p2}, \delta_{p3})' 3p \times 1.
\]

### 3.2 Specification of Covariances of Discount Functions

In (3.5) the specification of the covariance matrix of \( \eta \) is crucial since specifying the covariance structure of \( P = (P_1, \ldots, P_G)' \) or equivalently the covariance structure of \( \eta \) stochastically describes a structure of the joint realizations of \( G \) bond prices. In view of (3.2) the specification is directly related to that of the covariances of the stochastic discount factors \( D_g(s_{gi}) \) and \( D_h(s_{hm}) \) at each cash flow points \( s_{gi} \) and \( s_{hm} \) of the \( g \)-th and \( h \)-th bonds. We specify it as
\[
\text{Cov}(D_g(s_{gi}), D_h(s_{hm})) = \sigma^2 \lambda_{gh} f_{gh, jm},
\]
where \( \sigma^2 \) is a common variance and covariance factor that determines the size of variations, \( \lambda_{gh} \) is a covariance part related to the differences of maturities and \( f_{gh, jm} \) is another covariance part related to the difference of the cash flow points \( s_{gi} \) and \( s_{hm} \).

These two parts are further specified as
\[
\lambda_{gh} = \begin{cases} 
  e_{gg} & \text{if } g = h \\
  \rho e_{gh} & \text{if } g \neq h 
\end{cases}
\]
with \( e_{gh} = \exp(-\xi |s_{gm(g)} - s_{hm(h)}|) \) and
\[
f_{gh, jm} = \exp(-\theta |s_{gi} - s_{hm}|),
\]
where it is assumed that \( 0 \leq \theta, \rho \leq 1 \) and \( 0 \leq \xi \leq 2 \). These specifications imply:

1. as is expressed in \( e_{gg} \) of \( \lambda_{gg} \), the longer the maturity of each bond is, the larger the variance of each price is,
2. as is expressed in \( e_{gh} \) of \( \lambda_{gh} \), the larger the difference of the maturities of two bonds, the smaller the covariance is, and
(3) as is expressed in \( f_{gh, jm} \), the closer the two cash flow points are, the larger the covariance of the discount factors \( D_g(s_g) \) and \( D_h(s_{hm}) \) is.

Under this specification, the covariance matrix of \( \eta \) is given by

\[
\text{Cov}(\eta) = \text{Cov}(\eta_g, \eta_h) = \sigma^2 \lambda_{gh} \varphi_{gh} = \sigma^2 \Phi(\theta, \rho, \xi)
\]

with

\[
\varphi_{gh} = \sum_{j=1}^{M(g)} \sum_{m=1}^{M(m)} C_g(s_g) C_h(s_{hm}) f_{gh, jm}.
\]

3.3 Estimation of unknown parameters

As in Kariya and Kurata (2004), the unknown parameters are efficiently estimated by the GLS (generalized least squares) method, in which we minimize

\[
\psi(\beta, \theta, \rho) = \left[ \gamma - X \beta \right]' \Phi(\theta, \rho, \xi)^{-1} \left[ \gamma - X \beta \right].
\]

with respect to the unknown parameters. First, for given \((\theta, \rho, \xi)\), the minimizer of this function with respect to \( \beta \) is known to be the GLSE:

\[
\hat{\beta}(\theta, \rho, \xi) = \left( X' \Phi(\theta, \rho, \xi)^{-1} X \right)^{-1} X' \Phi(\theta, \rho, \xi)^{-1} \gamma
\]

and then the marginally minimized function \( \psi(\hat{\beta}, \theta, \rho, \xi) \) with substitution of \( \hat{\beta}(\theta, \rho, \xi) \) is minimized with respect to \((\theta, \rho, \xi)\), yielding the GLSE \( (\hat{\beta}, \hat{\theta}, \hat{\rho}, \hat{\xi}) \). In this paper to estimate \((\theta, \rho, \xi)\), we use the grid point method that maximizes the objective function for finitely split points or lattice of \((\theta, \rho, \xi)\) over the compact set \([0,0.9] \times [0,1] \times [0,2]\) with split unit 0.1.

4 Framework of Empirical Study

In this section we set up the framework of our empirical analysis. Recall that in (3.3) the four models M0, M1, M2, and M3 are defined: M0 with \((w_1, w_2, w_3) = (1,0,0)\) is the base case of no attribute effect, M1 with \((1,1,0)\) is M0 with maturity effect, M2 with \((1,0,1)\) is M0 with coupon effect, and M3 with \((1,1,1)\) is M0 with maturity and coupon effects.

The sample period for our analysis is five years from April 2006 through March 2011, where each analysis is made on the set of monthly USGB prices at the end of each month whose maturities are in the term interval between one year and 20 years. The reason why we do not use USGBs whose maturities are greater than 20 years is thin liquidity in terms of trading volumes, while the reason why we do not use USGBs whose
maturities are less than or equal to one year, which we call short-term USGBs, is due to
the following two points. First, it is well known that short-term USGBs are not well
priced in the market because banks have an incentive to hold USGBs to get a significant
fee for handling the redemption procedure for the Government. In other words, the
inclusion of the short-term USGBs in our analysis will make a distortion on the
polynomial coefficients and hence the corresponding will not well represent the market
movements of the term structures of interests. Second, short-term interest rates are
directly observed in markets though (risk-free) long-term interest rates whose
maturities are longer than one year are not readily available, except for swap rates that
carry bank credits. Our principal interest is to derive the movements of the term
structures of long-term (risk-free) interest rates.

The number of bond prices for each month varies from 121 to 185 as in Appendix
Tables A-1 and A-2, which respectively give the GLS-RSDs defined by \((\hat{\psi}/G)^{1/2}\) with
\(\hat{\psi}\) the monthly minimized GLS objective function and the simple residual standard
deviations (S-RSDs) of GLS residuals for our four models. Each set of monthly
cross-sectional USGBs contains high coupon bonds, which were issued as long term
bonds at the end of 80's and in the beginning of 90's. The coupon structure of our data is
implicitly shown in the graph of USGB prices in Fig 1, which plots realized prices and
M0 model prices of December 30, 2008, where M0 is estimated with \(p = 6\). Of course,
high prices in this graph correspond to high coupons, and as time moves month by
month, the high-coupon bonds in the upper left part of the graph disappear one by one.
This graph shows that as of December 30, 2008, our data include the prices of high
coupon bonds together with those of low coupon bonds up to the maturity terms of
about 13 years, implying that some possible coupon effects matter in M2 and M3 models.
The gaps between realized and model values, which are nothing but residuals, will be
analyzed in Section 7. It is noted that in terms of residuals, this graph is the worst case
with M0 GLS-RSD=0.00844 and M0 S-RSD=1.188 (dollar) at 08.12.30 where sample
size is 128 as in Table A-1 and Table A-2.
Because our sample period includes the Financial Crisis Period, we divide it into four sub-periods for a closer analysis. The division is based on a business cycle context and the information on the GLS-RSDs and S-RDSs of M0 model in Table A-1 and Table A-2.

Period I: 2006.4 ~ 2007.10, Period of Upturn Economy where M0 GLS-RSD < 0.001.
Period II: 2007.11 ~ 2008.7 Downturn Economy with the first subprime shock in 2008.3 where 0.001511 < M0 GLSV-RSD < 0.002996
Period III: 2008.8 ~ 2009.5 Financial Crisis Period including the Lehman Shock in 2008.9 where M0 GLS-RSD > 0.0026.
Period IV: 2009.6 ~ 2011.3 Post Financial Crisis Period M0 GLS-RSD < 0.0026

5 Model Selection for Polynomial Order
Now let us make a model selection with respect to the order $p$ of the polynomial in the mean discount function (3.3), which is based on (1) AIC-value, (2) GLS-RSD (GLS residual standard deviation) defined by $(\hat{\psi} / G)^{1/2}$ with $\hat{\psi}$ the minimized value of the GLS objective function, and (3) the simple residual standard deviation (S-RSD). Here the AICs are computed under normal distribution, though the normality assumption may not hold. This method may often be referred to as quasi-maximum likelihood method. In addition, we include the S-RSD because it gives values in dollar. In this section we will select 6 parameter model or equivalently 6 state variable (factor) model for M0.

The order selection in the mean discount function is based on M0 model, which is regarded as the base model for deriving term structures of interest rates as well as pricing USGBs. The other models that are of the same polynomial order as M0 are regarded as USGB-pricing models with additional explanatory power. The order of polynomials we aim to specify here is a common order for all the models over time. In
fact, using polynomials of a common order over time helps us study on dynamic changes of term structure of yields.

Fig 2 plots the 60 AIC, GLS-RSD and S-RSD values of M0 models estimated for each month for polynomial order $p = 2, 3, 4, 5, 6, 7, 8$. In view of our estimation procedure the GLS-RSD may be the one that we should use for model selection, but the other two also provide reference measures.

In each graph the case of $p = 2$ is the worst case and the cases $p = 3$ and $p = 4$ follow the worst. But the performances of the other cases are indistinguishable from the graphs. Compared to the AIC, the GLS-RSD distinguishes the models in the financial crisis period.

To select an order, we consider the mean AICs and GLS-RSDs of each models averaged over 60 months and plot them in Fig 3. The graph shows that the mean GLS-RSDs for $p = 6, 7, 8$ are almost same and indistinguishable though it is still decreasing toward $p = 8$. From the above observations in Fig 2 and from a parameter-saving viewpoint we judge that it will be proper to select $p = 6$ as the polynomial order of M0. A closer look at this problem in terms of S-RSDs is given to the appropriateness of this choice in the next section.
Fig 2 Comparisons of model performances of different specifications of the polynomial order $p = 2, 3, 4, 5, 6, 7, 8$ in the mean discount function of M0 model in terms of (1) AIC, (2) GLS-RSD and (3) S-RSD

Fig 3 The mean AICs (left) and mean GLS-RSDs (right) for M0 models of different polynomial orders

6 Comparisons of our Models with $p = 6$ and Overall Fitness
In this section we compare the models of M0 through M3. As measures for the fitness of the 4 models with \( p = 6 \) we also considered the AIC, the GLS-RSD \((\hat{\psi} / G)^{1/2}\) and the S-RSDs, all of which show similar movements over time though the physical units of the measures are different. The two graphs in Fig 4 are those of \((\hat{\psi} / G)^{1/2}\) and S-RSD.

![Graphs of GLS-RSDs and S-RSDs](image)

**Fig 4** Graphs of GLS-RSDs (above) of the minimized objective function and S-RSDs (below) of the four models over 60 months

It is clear that the GLS-RSDs in Fig 4 more differentiate M0 model from the other three models than the case of the S-RSDs, and that the M0 model is inferior to the other models except Period I. This features the model discriminating power of the GLS method that takes the correlation structure of data into account. In fact, as will be shown, the correlations of bond prices are increasing in these periods. Consequently in such an unstable period as the Financial Crisis Period the other three models are much better than M0 in pricing USGBs, which may reflect certain behaviors of investors who may prefer bond of certain maturities for liquidity or for asset-liability management or for both. In other words, in such a period attributes such as coupon and maturity matter in pricing USGBs. But they may not be important for deriving a term structure of interest rates, which will be discussed in Section 10. The significance of the differences
of the four models is discussed in the next section.

In the sequel, for easy understanding and interpretation we mainly use the S-RSDs as an overall evaluation of the model performances, since the ordering of the models in the case of GLS-RSD are the same as in the case of S-RSD. In fact, the S-RSD measures the performances of the models in terms of dollar and the information can be used for investment. In Table A-2 gives the residual standard deviations (S-RSDs) for the 4 models with \( p = 6 \) over 60 months and the information is summarized in Fig 4. From this graph it is observed that the S-RSDs of the 4 models have similar movements over time.

Fig 4 together with Table A-2 shows that the worst S-RSDs take place at 2008.12 of period III at the same time for all the models when it is in the middle of the Financial Crisis, and the S-RSD values are respectively 1.188, 0.769, 0.775 and 0.675 (dollar) for M0 through M3. And the second peak of S-RSDs take place in 2008.3 of period II for M0 (S-RSD=0.682), where 2008.3 is the month when the subprime problem that potentially existed first came out to surface as a shock of the defaults of some housing loan institutions. These S-RSDs in these worst cases will be still small enough to price USGBs on the average and to be able to use the term structure of interest rates implied by M0 even in the middle of such Financial Crisis, implying a robustness and an effectiveness of M0 model. In fact, as will be seen later, the estimated mean discount curves and hence yield curves via (3.3a) are relatively stable over time. In Section 7, we will see that the larger values of the S-RSDs except Period I seem to reflect certain behaviors seeking for liquidity. In fact, USGBs of 7-year or more maturities that are deliverable for USGB futures are preferred to the other ones because of the “cheapest” delivery. Hence the deviations from M0 model or equivalently residuals that led to the larger S-RSDs can be understood as results of panic or surprise behaviors for liquidity.

As has been observed above, all the S-RSDs are less than 1.188. To see the results in more detail, we summarize them for each period based on Table A-2. For Period I the S-RDSs are all less than 0.4 dollars for all the models, for Period II they are all less than 0.682 dollars for all the models, for Period III they are all less than 1.118 for all the models, and for Period IV they are all less than 0.7 for all the models. An overall summary for each period is given by Table A-1 with mean S-RSDs and standard deviations of S-RSDs.

These features are also observed in the criterion of GLS-RSD. In Table A-1 the GLS-RSD of each month is listed and in Table 6-1(1) the period-wise mean GLS-RSD with standard deviations are summarized. Though the physical unit is different from
that of S-RSD, the results are quite similar to the ones of the S-RSDs and indicate that 
M3 is the best. As has been pointed out, the GLS-RSD is of model discriminating 
power when the correlations of prices matter.

<table>
<thead>
<tr>
<th></th>
<th>M0</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
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<td></td>
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<td></td>
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</tr>
<tr>
<td>I</td>
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<tr>
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<tr>
<td></td>
<td>s 0.56804</td>
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<td>0.21977</td>
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<tr>
<td>I</td>
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Table 1 Means and Standard Deviations of (1) GLS-RSDs and (2) S-RSDs for each 
model for each period, where GLS-RSD values here are multiplied by 1000.

Among the four models the S-RSDs and GLS-RSDs of M3 model are the smallest, 
implying that the M3 model best fits USGB prices. But this does not necessarily mean 
that the M3 model is the best model to derive the implicit terms of structure of interest 
rates behind the prices, because the price approach that directly takes attributes into 
account is different from the interest approach that does not directly. This point will 
be discussed in Section 10, while the significance of the M3 model over the other models 
is verified in the next section.

7 Significance of maturity and coupon effects for pricing USGBs

This section investigates the pricing capability of each model for individual bonds in 
detail and tests the hypotheses of no maturity effect and/or no coupon effect. Here we 
distinguish the two model concepts:

(1) model for pricing USGB, which can be used for forming optimal portfolios of bonds 
or trading

(2) model for deriving term structures of interest rates, which can be used as market
“risk-free” rates for valuing many financial instruments and assets.

This section discusses on M1, M2, and M3 models in view of (1).

Fig 4 and Table 1 show the differences of performances of the 4 models. Apparently the M3 model is the best, M1 is the second, and M2 is the third, which correspond to additional explanatory power of bond attributes in the mean discount functions relative to the base M0 model. To measure the relative significance of the additional effects it is proper to use $F$-ratios based on the minimized values $\hat{\psi}$ of our objective function, where $F$-ratio measures a significance of the reduction of the quadratic sum of residuals (per parameter) due to the addition of explanatory attribute variables relative to the quadratic sum $\hat{\psi}$ ($QSR$) per degrees of freedom of the enlarged model: for example, in the comparison of M1 with M0, the $F$-ratio is defined as

$$F \text{ ratio } = \frac{[QSR(0) - QSR(1)]/#}{QSR(1)/df},$$

where # denotes the number of the incremental parameters due to the shift from M0 to M1 and $df$ the degrees of freedom of M1. This ratio may not be exactly distributed as $F$-distribution since the errors in (3.2) may not be normally distributed and since the quadratic forms $\hat{\psi}$ are not quadratic in error terms. But $F$-ratio itself is a measure of the degree of incremental explanatory power, and so we say that model 1 is more significant than model 0 if the above $F$-ratio is greater than 2, or equivalently if

$$[QSR(0) - QSR(1)]/# > 2QSR(1)/df.$$  

Fig 5 plots the $F$-ratios of (a) M0 vs M1, (b) M0 vs M2, and (c) M1 vs M3, where $F$-ratios are cut off at 20 in the graphs if they are greater than 20. From Fig 5 (a) it is observed that M1 is far more significant than M0 for all the months, implying the significance of the maturity effect in pricing USGBs. On the other hand, a comparison between (a) and (b) makes us observe that relative to M0, the coupon effect in M2 model is a little less significant than the maturity effect, though it is very significant for all the months. The lower $F$-ratios of the coupon effect in financial crisis period may be interpreted as liquidity preference over high coupon preference. The liquidity preference will be associated with maturity effect later. The graph (c) compares M1 with M3 and shows that M3 model is more significant than M1, though the incremental explanatory power of coupon effect in addition to maturity effect is less than the one obtained by shifting from M0 to M1 or M2. And it is interestingly observed that the coupon effect additional to maturity effect is getting larger in post-crisis period, implying that high coupon bonds are more preferred together with maturity in this period. Consequently the M3 model is significantly the best in pricing USGBs and can be used for pricing USGBs, constructing an optimal portfolio of USGBs and trading USGBs for returns, though it may not be proper to derive a “risk-free” yield curve..
The yield curves of these models are affected by their attributes and methods of their derivation will be discussed in Section 10.

![Graphs of F-ratios](image)

**Fig 5** Graphs of F-ratios: (a) M0 vs M1, (b) M0 vs M2, (c) M1 vs M3

8 **Parameter Changes**

As has been shown in Table 1 or Table A-2, each model with the cross-sectionally estimated parameters well price each monthly set of bonds simultaneously. However the cross-sectional parameters change over time. This implies that while a small number of parameters stably command a large number of bond prices, the parameters are time-varying probably because investors are sensitive to cross-sectionally rebalancing or optimizing or arbitrage opportunities in each changing environment when prices are formed simultaneously in the market. Therefore the parameters change
dynamically along with evolutions of environments. This point poses an important question as to whether parameters involved in modeling financial data are in fact constant over time even if environments are changing and investors are corresponding to changes. In this section, we observe how the estimated parameters of M0 model change in time though we do not model them as time series in this paper.

Let the estimated parameters of M0 be $\beta_i$ $(i = 1, \cdots , 6)$ where $\beta_i$ corresponds to the $i$-th order of the 6-th order polynomial or equivalently 6 factor model. Fig 6 plots the M0 parameter movements over time where the 3 graphs are of different scales and Table 2 gives the correlation matrix of the parameters. From the top graph it is found that $\beta_1$ and $\beta_2$ are of the same order $10^{-2}$ as they stand though $\beta_2$ is multiplied by $s^2$ in the discount function while $\beta_1$ is simply multiplied by $s$. This implies that $\beta_1 s + \beta_2 s^2$ is a leading factor to form a decreasing pattern of the discount function. It is interesting to observe that $\beta_1$ is crossing $\beta_2$ from minus to plus in 2008.12. Before the date the downward trend of discount functions is made by the larger negative values of $\beta_1 s$ but after 2008.12 the values of $\beta_1$ and $\beta_2$ are almost symmetric about 0 and the downward trend is basically made by the negative values of $\beta_2 s^2$. In the middle graph $\beta_3$ and $10\beta_3$ are shown to be of the same order $10^{-3}$ and they are almost symmetric about 0, implying that $\beta_3 s^3 + \beta_4 s^4$ becomes near 0 around $s=10$ and $\beta_3 s^4$ gradually dominates $\beta_3 s^3$ after $s=10$. A similar pattern is observed in the bottom graph of $(\beta_5, 100\beta_6)$, where in this case $\beta_5$ and $\beta_6$ are respectively of order $10^{-6}$ and $10^{-8}$. Also the width of the two graphs in each figure is shown to commonly get larger after 2008.10, which will be due to the financial crisis. Overall the 6 parameters are synchronizing in motion and the movements will give us some more information when we consider macroeconomic factors together with them, though the research problem is left open.
Fig 7 Changes of M0 Coefficients over 60 Months.

Table 2 Correlation Matrix of Coefficients

<table>
<thead>
<tr>
<th></th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_4$</th>
<th>$\beta_5$</th>
<th>$\beta_6$</th>
</tr>
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<tbody>
<tr>
<td>$\beta_1$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>1.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.453</td>
<td>-0.837</td>
<td>1.000</td>
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<td></td>
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<tr>
<td>$\beta_4$</td>
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<td>1.000</td>
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</tr>
<tr>
<td>$\beta_5$</td>
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<td>0.835</td>
<td>-0.971</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>-0.098</td>
<td>0.404</td>
<td>-0.744</td>
<td>0.915</td>
<td>-0.984</td>
<td>1.000</td>
</tr>
</tbody>
</table>

On the other hand, the graphs of $\beta_2$ and $\beta_4$ move reversely, and so do the graphs of $\beta_3$ and $\beta_5$. In fact, Table 2 shows that $\beta_2$, $\beta_3$, $\beta_4$, $\beta_5$ and $\beta_6$ are all highly correlated and two adjacent pair has high negative correlations of less than -0.837, and that $(\beta_2, \beta_4)$, $(\beta_3, \beta_5)$ and $(\beta_4, \beta_6)$ are of high positive correlations of more than
Further $\left( \beta_1, \beta_3 \right)$ and $\left( \beta_3, \beta_5 \right)$ are negatively and highly correlated and these alternate sign change and high correlation structure makes a balance and stability to describe term structure of implied interest rates via the discount function.

Next, let us consider the changes of the parameters in the covariance matrix. In all the models the parameter $\theta$ is estimated as 0 for all the 60 monthly sets of data, implying that

$$f_{gh, jm} = \exp(-\theta |s_{gh} - s_{hm}|) = 1 \text{ in (3.7) and so } \varphi_{gh} = \sum_{j=1}^{M(g)} \sum_{m=1}^{M(m)} C_g(s_{gh})C_h(s_{hm})$$

in $\text{Cov}(P_g, P_h) = \sigma^2 \lambda_{gh} \varphi_{gh}$. Therefore the variance of $P_g$ is given by

$$\text{Var}(P_g) = \sigma^2 \left[ \sum_{j=1}^{M(g)} C_g(s_{gh}) \right]^2$$

with $\sigma^2 = \psi / G$.

Here we note that in the case of M0 model $\sqrt{\psi / G}$ is uniformly less than 0.0011 in Period I and it is less than 0.0030 in Period II as in Fig 4. And so when $\sqrt{\psi / G} = 0.001$ dollars and $c_g = 2$ dollars with $s_{M(\psi)} = 5$ or 10 or 20 years where $c_g$ is the size of coupon of $g$, the standard deviation

$$(8.1) \quad |\text{Var}(P_g)|^{1/2} = \sqrt{\psi / G} \sum_{j=1}^{M(g)} C_g(s_{gh}) = \sqrt{\psi / G} \times 100 \times \sqrt{\psi / G} \times c_g \times s_{M(\psi)}$$

is respectively $0.001 \times 110 = 0.11$ or $0.001 \times 120 = 0.12$ or $0.001 \times 140 = 0.14$. The larger the coupon rate is, the larger the standard deviation. As in Appendix Table A-1, the maximum of $\sqrt{\psi / G}$ attains at 2008.12 and it is 0.008436, in which case the above case becomes 0.928, 1.012 or 1.181 if $c_g = 2$, implying the unconditional volatilities of the bond prices are about 1% with respect to the face value = 100 dollars. If $(c_g, s_{M(\psi)}) = (5, 10)$, $(7, 15)$ and $(7, 20)$, the volatilities are respectively 1.434, 1.729 and 2.025%, which is large.

Similarly for each coupon rate $c_g$, standard deviations of prices are evaluated.

On the other hand, the correlation of two bond prices $(P_g, P_h)$ is given by

$$(8.2) \quad \text{Correl}(P_g, P_h) = \lambda_{gh} = \rho \exp(-\frac{\psi}{G} |s_{M(\psi)} - s_{M(h)}|) \quad (g \neq h).$$

Here the overall common parameter $\rho$ that dominates these correlations for any
$g$ and $h$ is estimated in Fig 7 where the upper graph is the graph of $\rho$’s in M0 and the lower is the graph of $\rho$’s in M3. In M3 the correlations get larger on the subprime shock in March, 2008 and the Lehman Shock in September, 2008. In addition, the correlations of M3 are relatively larger than those of M0. Another parameter $\xi$ to determine $\text{Correl}(P_g, P_h)$ is estimated in Fig 8, from which $0.1 \leq \xi \leq 4.0$ for M0 and M3 follow. Interestingly $\xi$’s are decreasing in Periods II, III and IV, implying that the correlations are increasing in these periods, when $|s_{M(g)} - s_{M(h)}|$ is given. However, $\text{Correl}(P_g, P_h)$ rapidly decreases as $|s_{M(g)} - s_{M(h)}|$ increases. This may be due to our specification of the covariance matrix with correlations exponentially decreasing in $\xi |s_{M(g)} - s_{M(h)}|$. In 2008.12 our empirical results shows small correlations in large maturities and relatively larger. A better specification may be required. This problem is left open here.

![Fig 7 Changes of $\rho$’s over 60 months. The $\rho$’s of M0 are in the upper graph and the $\rho$’s of M3 are in the lower graph.](image-url)
9 Individual Price Fitness and Influences of the Financial Crisis

In this section, we investigate on time series variations of the cross-sectional term structures of about 140 residuals over 60 months and discuss about some influences of the Financial Crisis on the performances of our models. The fitness of our models to individual prices appears in the individual residuals of the models.

As has been seen in Fig 1, the realized and M0 values of December 30, 2008, which is the worst case in the S-RSDs and the GLS-RSD, are plotted, where sample size is 128. The gaps between the realized and model values in the worst case are plotted as residuals in Fig 9. From this residual term structure it is found that the residuals of M0 model are very large and lie in the interval [-4.5, 5] where the S-RSD of M0 model is 1.188 (see Table A-2 in Appendix) and large residuals are rather similar to the coupon structure of the USGBs at the time exhibited in Fig 1. In other words, investors seem to have preferred high coupon bonds in the crisis. In addition they may have preferred bonds of maturities 7 through 10 years for preparing settlement of USGB futures for long and short positions. On the other hand, the case of M3 model in the second graph of Fig 9 does not exhibit these patterns related to the attributes and all the residuals sit in the range [-1.8, 1.9]. where the S-RSD of M0 model is 0.675 (see Table A-2 in
Appendix). Note that in Table 1, the standard deviations are summarized in each period of I-IV.

Fig 9 Term Structure of M0 and M3 Residuals of 2008.12.30

For comparison, the best case in M0 model is the case of January 31, 2007 in Period I of upturn economy with GLS-RSD 0.000420, S-RSD 0.262 and sample size 121 as in Tables A-1 and A-2. As in Fig 10, the term structure of M0-model residuals is rather flat around 0 dollar and does not show a systematic pattern. The residuals that are greater than 0.1 in absolute values are the following 13 residuals with maximum 0.325 in absolute value

0.129, 0.134, 0.104, -0.148, -0.113, -0.126, 0.133, 0.116, 0.168, 0.325, -0.114,
-0.130, -0.136,

implying that all the residuals except the one with 0.325 are less than 0.15 dollars in absolute values and M0 fits individual prices almost perfectly in such a good economic period as Period I that includes January 31, 2007, where the S-RSDs of M0 and M3 are respectively 0.262 and 0.182 (see Appendix Table A-2). The mean S-RSD of Period I is only 0.308 with standard deviation 0.0358. The bonds with these residuals flatly spread in maturity over a wide range.

Also the residuals of M3 are plotted in Fig 10 and except for a few they are all in the area between -0.1 and 0.1. Comparing this best case with the worst case, it is
found that M3 is the best in pricing individual USGBs though M0 is also good enough as a pricing model.

Fig 10 Term Structure of M0 and M3 Residuals of 2007.1.31

10 Term structures of discount curves and yield curves and Financial Crisis

As has been shown above, M1, M2 and M3 models value USGBs better than M0. In this section we consider the effectiveness of our models in terms of the term structures of implied interest curves implied by the term structures of discount curves, which are given via the formal transformation:

\[
R_j(s : 1, z_{j2}, z_{j3}) = -\frac{1}{s} \log D_j(s : 1, z_{j2}, z_{j3})
\]

as in (3.4), where \( j = 0, 1, 2 \), and 3 respectively correspond to M0, M1, M2 and M3 models. First we shall discuss about how the term structure functions in (10.1) are defined for M1, M2 and M3 and then consider their effectiveness in relation to the effectiveness of M0 term structure function.

In the case of M0 it is obvious that functions \( \overline{D}_0(s) \equiv \overline{D}_0(s : 1, 0, 0) \) and \( R_0(s) \equiv R_0(s : 1, 0, 0) \) defined on \( 1 \leq s \leq 20 \) are respectively discount rate curve and term structure of implied interest rates. On the other hand, although in estimating the parameters of M1, M2 and M3 models, \( (z_{j2}, z_{j3}) \)'s were set by \( (z_{20}, z_{30}) = (0, 0) \), \( (z_{21}, z_{31}) = (s_{M(g)}, 0) \), \( (z_{22}, z_{32}) = (0, c_g) \) and \( (z_{23}, z_{33}) = (s_{M(g)}, c_g) \) with \( s_{M(g)} \) maturity and \( c_g \) coupon for bond \( g \), the discount functions implied by these models are not those
of (10.1) with these \((z_{2j}, z_{3j})\)'s. Here to define a discount function in the case of M1 which is free from the attributes, note that if bond \(g\) is a zero-coupon bond with maturity \(s_{M(s)}\), the discount rate will be given by \(s = s_{M(s)}\). Hence it is proper to define the discount function of M1 by \(\bar{D}_1(s) \equiv \bar{D}_1(s:1,s,0)\), which is nothing but the discount rate for a zero-coupon bond of maturity \(s\) under M1. This in turn defines the term structure function by \(R_i(s) \equiv R_i(s:1,s,0)\) by (10.1). In the case of M2, the discount function is obtained by setting \(c_s = 0\) with \(\bar{D}_2(s) \equiv \bar{D}_2(s:1,0,0)\), which is also the discount rate for a zero-coupon bond of maturity \(s\) under M2, and hence \(R_2(s) \equiv R_2(s:1,0,0)\). Similarly in the case of M3 we define \(\bar{D}_3(s) \equiv \bar{D}_3(s:1,s,0)\) and \(R_3(s) \equiv R_3(s:1,s,0)\). These 4 discount functions have different parameters estimated under different specifications. A basic question to pose here is whether these three functions \(\bar{D}_i(s), i = 1, 2, 3\) perform as discount functions implying term structures of interest rates. This question is positively answered for M1 and M2 in terms of term structure function of interest rates. But in the case of M3 the term structure of interest rates is different from that of M0 as in Fig 11 (c).

Assuming that \(R_0(s)\) effectively represents its term structure, Fig 11 plots (a) the graphs of \(R_0(s)\) and \(R_1(s)\) and (b) the graphs of \(R_1(s)\) and \(R_2(s)\) and (c) the graphs of \(R_2(s)\) and \(R_3(s)\) for the month of 2007.1. Comparing these three graphs will find that \(R_0(s)\) is relatively larger than \(R_j(s) (j = 1, 2, 3)\) in the range \([9.4, 17.3]\) for \(s\), implying that the shape of the term structure of interest rates of \(R_0(s)\) in this range is too high relative to the cases where the attribute effects are taken into account. The fact that differences \(R_i(s) - R_0(s)\) of interest rates in this range are ordered as 3, 1, 2 will imply that the maturity effect is larger than the coupon effect and the combined attributes make the largest effect on the term structures, where this ordering of the attribute effects has been observed in the \(F\)-ratios for pricing USGBs. On the other hand after the range \([17.3, 20]\) for \(s\), \(R_i(s)\) behaves differently from \(R_1(s)\) and \(R_2(s)\) and it may pose a question on the effectiveness of the definition of \(\bar{D}_3(s) \equiv \bar{D}_3(s:1,s,0)\). In fact, in the case of M3, \(R_3(s)\) does not behave like \(R_0(s)\) in terms longer than 17.5 years. This will be probably because maturity effect and coupon effect are mutually
dependent and the estimated parameters reflect the dependency so that setting $\tilde{D}_j(s) \equiv D_j(s;1,s,0)$ may not work for more than 15 years where the order 6 of the polynomial with the setting makes the roles of the parameters unbalanced by setting coupon equal to 0. Therefore although M3 is the best in pricing USGBs, it will not be appropriate in deriving term structures. We will do more research elsewhere. We leave this question open.

It is noted that these observations on the shape of the term structures are common to other time points, though the differences $R_j(s) - R_0(s)$ widen in Financial Crisis period and ranges where the differences become positive or negative moves.

Fig 11 Term Structures of (a) $R_0(s)$ and $R_1(s)$, (b) $R_0(s)$ and $R_1(s)$ and (c) $R_0(s)$ and $R_3(s)$, where the time is 2007.1.

10-1 Term structures of discount rates and interest rates

Now let us consider an effectiveness of the M0 term structure of interest rates. In Fig
12(a) a 3D graph of the 60 term structures of discount rates given by $\tilde{D}_0(s)$ is drawn, which is in one-one correspondence with the 3D graph of term structures of interest rates given in Fig 12(b). Therefore the effectiveness of $\tilde{D}_0(s)$ is verified by verifying the effectiveness or capability of $R_0(s)$ as the term structure of interest rates for each month. If it is effective, then it turns out that the model is effective or capable to derive the term structures of interest rates. Hence we look into changes of the term structures in Fig 12(b) and compare them (plain vanilla) dollar-dollar swap rates that represent directly traded market rates of interest rates.

(a) $\tilde{D}_0(s)$

(b) $R_0(s)$

Fig 12 Term Structure Movements of (1) Discount Rates and (2) Implied Interest Rates.
In Fig 12(a)(b), the left axis and right axis respectively represent the historical time and the term to maturity, and at 33 time point (December 2008), there is a series of mountains along the right axis, showing volatile variations of the subprime shock. It seems that the movement in implied interest rates (2) well represents an overall movement of the term structures. This observation will be partly justified in the next subsection.

10.2 Comparisons of the derived interest rates with swap rates.

In Fig 13(a), for each rates of 3-year, 5-year, 7-year, 10-year, 15-year and 20-year maturities the time series movements of swap rate and M0 implied rate are plotted. It is clear that they are all very well synchronized and hence the M0 implied bond rates are judged to behave properly. In the 3-year and 5-year rates swap rates are uniformly larger than the bond rates. Looking into the graphs of the differences of the two rates in Fig 13(b) that are regarded as credit spreads of swap rates relative to USGB bond rates, the gap in 3-year rates is as a whole increasing over the whole period, which implies that in 3 year maturity the credit quality of USGB rates is relatively increasing in this period with the gap at 2011.3 being 18bpt, while the gap in 5 year rates drops down abruptly after 2008.08 and attains minimum at 2010.10 with positive gap. In these rates in the Post Financial Crisis the gaps between the two rates are slightly larger than those of Period I, which may reflect a fact that in these time horizons investors concern about the credit quality of downgraded financial institutions in private sector. On the other hand, in the terms greater than 5 years swap rates go down below bond rates at some points, which may imply that investors worry about the possibility of defaults of USGBs in such time horizons in association with the huge deficit of the Government budget. Compared with gaps in 5-year rates, those in 7-year rates slightly decrease after the Financial Crisis but still remain positive values.

However, in 10-year rates the first cross of the swap rate over the bond rate takes place in 2008.10 and the bond rate almost is equal to the swap rate after Crisis, implying that the credit quality of 10 year USGBs is valued much the same as the swap rate in this period. And even in Post Crisis Period the credit quality has not been retrieved as much as in Period I and II.

This feature becomes stronger in 15 year and 20 year rates. In these rates the first cross takes place at the same time after the Lehman Shock of 2008.9 and the gaps consistently are negative value. The maximum spread with the swap rate being lower than the bond rate is surprisingly over 50bpt in these 15-year and 20-year rates.
Fig 13(a) Time Series of Swap rates and M0 implied bond rates
Fig 13(b) Time Series of Spreads defined by Swap Rate minus M0 implied Bond Rate

As a final remark, it is pointed out that these empirical relations between swap rates and bond rates fail to claim the validity of the relation between swap rates and the theoretical swap pricing formula: 

\[ \text{swap rate} = 100 \{ 1 - \frac{D_0(s_m)}{0.5 \sum_{j=1}^{M} D_0(s_j)} \} \]

where \( s_1, s_2, \ldots, s_M \) are the times at which swaps are made. But we do not elaborate to verify it here.

11 Conclusion

In this paper, expressing discount function by a version of forward interest rates that may depend on bond attributes, we first verified the theoretical consistency and validity of our discount function approach to pricing bonds and deriving term structures of interest rates and our cross-sectional models with different correlations among GB prices are likely to be more robust as we can avoid the specification of an interest rate model. Based on a comprehensive empirical study in the GLS framework, we proposed the M0 model as a base model and M3 model with polynomial of order 6 as the discount functions in pricing USGBs for all the times. But in describing the term structure of implied interest rates, the M0 was shown to be empirically effective even in the Financial Crisis Period and can be used for various purposes. Also maturity-adjusted and coupon-adjusted term structures of interest rates were derived. The standard deviations and correlations of individual USGBs were enabled to derive in our GLS framework. In addition, we observed that the credit spreads of swap rate minus bond rate became negative in the USGBs of longer maturity since October 2008, which may reflect the budget problem of the Government.

There remain the two important problems left out in this paper: the problem of alternative specifications of the covariance matrix of GB prices and the problem of making our cross-sectional models dynamic. These problems will be treated elsewhere.

References


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### Appendix

#### Table A-1: GLS-RSD × 1000

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| 06/04| 110 | 0.280| 0.195| 0.209| 0.181| 08/10| 127 | 1.116| 0.713| 0.814| 0.645|
| 06/05| 111 | 0.285| 0.221| 0.221| 0.202| 08/11| 127 | 1.159| 0.735| 0.764| 0.673|
| 06/06| 113 | 0.291| 0.226| 0.228| 0.219| 08/12| 128 | 1.188| 0.769| 0.775| 0.675|
| 06/07| 114 | 0.264| 0.214| 0.220| 0.208| 09/01| 129 | 1.138| 0.746| 0.767| 0.662|
| 06/08| 115 | 0.306| 0.231| 0.255| 0.219| 09/02| 130 | 1.060| 0.657| 0.674| 0.617|
| 06/09| 114 | 0.298| 0.218| 0.238| 0.203| 09/03| 135 | 0.879| 0.577| 0.581| 0.549|
| 06/10| 117 | 0.283| 0.217| 0.237| 0.194| 09/04| 137 | 0.796| 0.518| 0.530| 0.501|
| 06/11| 120 | 0.291| 0.236| 0.250| 0.209| 09/05| 136 | 0.756| 0.451| 0.516| 0.436|
| 06/12| 119 | 0.296| 0.225| 0.255| 0.205| 09/06| 141 | 0.694| 0.448| 0.502| 0.428|
| 07/01| 121 | 0.262| 0.194| 0.225| 0.182| 09/07| 143 | 0.665| 0.399| 0.452| 0.386|
| 07/02| 123 | 0.295| 0.221| 0.256| 0.210| 09/08| 145 | 0.659| 0.427| 0.459| 0.379|
| 07/03| 122 | 0.321| 0.236| 0.280| 0.225| 09/09| 148 | 0.596| 0.389| 0.432| 0.338|
| 07/04| 123 | 0.317| 0.236| 0.277| 0.223| 09/10| 147 | 0.546| 0.364| 0.428| 0.335|
| 07/05| 125 | 0.286| 0.231| 0.242| 0.218| 09/11| 153 | 0.547| 0.369| 0.409| 0.342|
| 07/06| 124 | 0.300| 0.251| 0.255| 0.248| 09/12| 152 | 0.482| 0.332| 0.413| 0.290|
| 07/07| 127 | 0.321| 0.240| 0.272| 0.234| 10/01| 154 | 0.484| 0.325| 0.393| 0.298|
| 07/08| 128 | 0.396| 0.280| 0.288| 0.265| 10/02| 157 | 0.488| 0.334| 0.390| 0.307|
| 07/09| 126 | 0.371| 0.264| 0.271| 0.246| 10/03| 162 | 0.432| 0.324| 0.340| 0.285|
| 07/10| 128 | 0.367| 0.272| 0.289| 0.257| 10/04| 164 | 0.417| 0.324| 0.334| 0.297|
| 07/11| 128 | 0.488| 0.355| 0.393| 0.328| 10/05| 165 | 0.515| 0.429| 0.425| 0.418|
| 07/12| 126 | 0.489| 0.348| 0.386| 0.319| 10/06| 170 | 0.518| 0.450| 0.449| 0.440|
| 08/01| 128 | 0.511| 0.334| 0.390| 0.305| 10/07| 169 | 0.542| 0.485| 0.491| 0.488|
| 08/02| 128 | 0.584| 0.373| 0.429| 0.322| 10/08| 174 | 0.472| 0.413| 0.411| 0.412|
| 08/03| 128 | 0.682| 0.499| 0.500| 0.485| 10/09| 176 | 0.421| 0.344| 0.347| 0.339|
| 08/04| 128 | 0.578| 0.415| 0.419| 0.391| 10/10| 175 | 0.423| 0.307| 0.329| 0.301|
| 08/05| 125 | 0.572| 0.385| 0.390| 0.346| 10/11| 180 | 0.417| 0.315| 0.338| 0.292|
| 08/06| 127 | 0.626| 0.405| 0.411| 0.362| 10/12| 178 | 0.486| 0.372| 0.396| 0.351|
| 08/07| 127 | 0.612| 0.401| 0.407| 0.359| 11/01| 182 | 0.487| 0.370| 0.430| 0.360|
| 08/08| 125 | 0.643| 0.407| 0.410| 0.376| 11/02| 184 | 0.431| 0.342| 0.391| 0.337|
| 08/09| 127 | 0.726| 0.546| 0.547| 0.517| 11/03| 181 | 0.411| 0.322| 0.374| 0.310|

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