Single-name concentration risk in credit portfolios: a comparison of concentration indexes

Raffaella Calabrese, Francesco Porro

1 Introduction

The Asymptotic Single-Risk Factor (ASRF) model (Gordy, 2003) that underpins the Internal Rating Based (IRB) approach in the Basel II Accord (Basel Committee on Banking Supervision, BCBS 2004) assumes that idiosyncratic risk has been diversified away fully in the portfolio, so that economic capital depends only on systematic risk contributions. Systematic risk represents the effect of unexpected changes in macroeconomic and financial market conditions on the performance of borrowers. On the other hand, idiosyncratic risk represents the effects of risks that are particular to individual borrowers. In order to include idiosyncratic risk in economic capital, Basel II (BCBS, 2004) requires that banks estimate concentration risk. Concentration risks in credit portfolios arise from an unequal distribution of loans to single borrowers (single-name concentration) or industrial or regional sectors (sector concentration). This paper is focused only on the single-name concentration, in particular in the context of loan portfolios.

Five concentration measures, which were proposed in welfare and monopoly theory, are compared as regards six desiderate properties for measurements of concentration risk. A widely applied index to measure concentration risk is Gini index that measures inequality rather than concentration. To understand the difference between inequality and concentration, a portfolio that contains few large exposures is considered. By including in such portfolio many small exposures so that even in aggregate their share of the portfolio exposure is very low. Therefore, concentration has not been significantly affected, but the degree of inequality in borrowers’ exposures has greatly increased. It follows that inequality measures are very sensitive to the number of small exposures.

Raffaella Calabrese  
University College Dublin, e-mail: raffaella.calabrese@ucd.ie

Francesco Porro  
Università degli Studi di Milano-Bicocca, e-mail: francesco.porro1@unimib.it
The other indexes (Hannah-Kay index, Herfindahl-Hirschman index, Theil index, Hall-Tidemann index) were proposed in monopoly theory. Concentration risk, analogously to industrial concentration, is due to both a small number of loans in portfolio and a small number of loans represents a high share of portfolio exposure. It follows that these indexes, unlike Gini index, depend on the number of loans in portfolio. By normalizing the above-mentioned indexes, this important information would be lost, so the normalization is not applied to them in this paper.

An interesting result of this work is that the above-mentioned concentration indexes satisfied all the six desirable properties for measurements of concentration risk, unlike Gini and Theil indexes. In order to compare the properties of the concentration indexes that are not established by the six properties, six portfolios with different concentration levels are analyzed. Among them, the portfolio with the highest concentration risk is defined compliant to the regulation of the Bank of Italy (Banca d’Italia, 2006). In the six portfolios both the total exposure and the number of loans are changed in order to analyse their impact on the concentration indexes. The main results of this numerical application are that the Reciprocal of Hannah-Kay index with $\alpha = 3$, the Herfindahl-Hirschman index and the Hall-Tidemann index stress the impact of lager exposures. On the contrary, the Reciprocal of Hannah-Kay index with $\alpha = 0.5$ and the Theil index underline the importance of smaller exposures.

The paper is organized as follows. Section 2 defines the six properties of a single-name concentration index and the relationships among them. Section 3 investigates which properties are satisfied by Gini index and section 4 by the indexes for the industrial concentration. In section 5 those concentration indexes are used to measure the concentration of six portfolios with different levels of concentration. Section 6 is devoted to conclusions and future developments.

2 Properties of a single-name concentration index

Consider a portfolio of $n$ loans. The exposure of the loan $i$ is represented by $x_i \geq 0$ and the total exposure of the portfolio is $\sum_{i=1}^{n} x_i = T$. In the following, a portfolio is denoted by the vector of the shares of the amounts of the loans $s = (s_1, s_2, \ldots, s_n)$: the share $s_i \geq 0$ of loan $i$ is defined as $s_i = x_i / T$. Whenever the shares of the portfolio $s$ need to be ordered, the corresponding portfolio obtained by the increasing ranking of the shares will be denoted by $s_{(\cdot)} = (s_{(1)}, \ldots, s_{(n)})$. It is clear that any reasonable concentration measure $C$ has to satisfy $C_n(s) = C_n(s_{(\cdot)})$. The following six properties are well-known in the literature as the most desirable ones that a single-name concentration measure $C$ should satisfy. Indeed they were born in the industrial concentration framework, nevertheless their translation to credit analysis can be considered successful (cfr [3], [7] and [15]). In the following, where necessary, in order to remark the number $n$ of the loans in the portfolio, the single-name concentration measure will be denoted with $C_n$.

1. (Transfer principle) The reduction of a loan exposure and an equal increase of a bigger loan must not decrease the concentration measure.
Let \( s = (s_1, s_2, \ldots, s_n) \) and \( s^* = (s_1^*, s_2^*, \ldots, s_n^*) \) be two portfolios such that

\[
\begin{align*}
    s_i^* &= \begin{cases} 
        s_i - h & \text{if } k = i \\
        s_j + h & \text{if } k = j \\
        s_k & \text{otherwise},
    \end{cases}
\end{align*}
\]

where \( s_i < s_j \) and \( h > 0 \). Then \( C(s) \leq C(s^*) \)

2. **Uniform distribution principle** The measure of concentration attains its minimum value, when all loans are of equal size.

Let \( s = (s_1, s_2, \ldots, s_n) \) be a portfolio of \( n \) loans. Then \( C(s) \geq C(s_e) \), where \( s_e \) is the portfolio with equal-size loans, that is \( s_e = (1/n, \ldots, 1/n) \).

3. **Lorenz-criterion** If two portfolios, which are composed of the same number of loans, satisfy that the aggregate size of the \( k \) biggest loans of the first portfolio is greater or equal to the size of the \( k \) biggest loans in the second portfolio for \( 1 \leq k \leq n \), then the same inequality must hold between the measures of concentration in the two portfolios.

Let \( s = (s_1, s_2, \ldots, s_n) \) and \( s^* = (s_1^*, s_2^*, \ldots, s_n^*) \) be two portfolios, both of them with \( n \) loans. Let \( s(i) = (s(1), s(2), \ldots, s(n)) \) and \( s^*(i) = (s^*(1), s^*(2), \ldots, s^*(n)) \) denote the two portfolios with shares in not-decreasing order (it means that if \( i < j \), then \( s(i) \leq s(j) \) and \( s^*(i) \leq s^*(j) \)). If \( \sum_{i=k}^n s^*_i \geq \frac{n}{m} \sum_{i=k}^n s(i) \) for \( k = 1, \ldots, n \), then \( C(s) \leq C(s^*) \).

4. **Superadditivity** If two or more loans are merged, the measure of concentration must not decrease.

Let \( s = (s_1, \ldots, s_l, \ldots, s_j, \ldots, s_n) \) be a portfolio of \( n \) loans, and \( s^* = (s_1, \ldots, s_{l-1}, s_{l+1}, \ldots, s_{j-1}, s_{m}, s_{j+1}, \ldots, s_n) \) a portfolio of \( n-1 \) loans such that \( s_m = s_l + s_j \). Then \( C_n(s) \leq C_{n-1}(s^*) \).

5. **Independence of loan quantity** Consider a portfolio consisting of loans of equal size. The measure of concentration must not increase with an increase in the number of loans.

Let \( s_{e,n} = (1/n, \ldots, 1/n) \) and \( s_{e,m} = (1/m, \ldots, 1/m) \) be two portfolios with equal-size loans and \( n \geq m \), then \( C_n(s_{e,n}) \leq C_m(s_{e,m}) \).

6. **Irrelevance of small exposures** Granting an additional loan of a relatively low amount does not increase the concentration measure. More formally, if \( s' \) denotes a certain percentage of the total exposure and a new loan with a share of \( s' \) of the total exposure is granted, then the concentration measure does not increase.

Let \( s = (s_1, s_2, \ldots, s_n) \) be a portfolio of \( n \) loans, where \( s_i = x_i/T \). Then, there exists \( s' \) such that for all \( \tilde{s} = \tilde{x}/(T + \tilde{x}) \leq s' \), considering the portfolio of \( n+1 \) loans \( s^* = (s_1^*, s_2^*, \ldots, s_{n+1}^*) \) with shares given by:

\[
    s_i^* = \begin{cases} 
        x_i/(T + \tilde{x}) & \text{if } i = 1, 2, \ldots, n \\
        \tilde{x}/(T + \tilde{x}) & \text{if } i = n+1
    \end{cases}
\]

it holds that \( C(s) \geq C(s^*) \).

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has the following shares

\[ s^*_i = \begin{cases} 
\frac{x_i}{T + \bar{x}} & i = 1, 2, \ldots, n \\
\frac{\bar{x}}{T + \bar{x}} & i = n + 1.
\end{cases} \]

It holds that \( C(s) \geq C(s^*) \).

A few remarks on the aforementioned properties can be useful. The first three properties arise from the framework of concentration analysis of income distribution: they are reasonable requirements that an inequality index wishes to satisfy. In these properties the number \( n \) of the loans of the portfolio does not change, while in the other ones \( n \) changes. This happens because properties 4, 5 and 6 point out the importance of the number of the loans in a portfolio. The principle of transfer (property 1 is very classical: it has been introduced for the first time by Pigou (in 1912, see [16]) and Dalton (in 1920 see [5]) at the beginning of the last century. Also property 3 comes from the past: it is related to the Lorenz curve, a milestone in the study of the income concentration, introduced by Lorenz in 1905 (see [14]). Property 4 can be applied more than one time, therefore the concentration must not decrease also in case of a merge of three or more loans. The properties 4 and 5 come from the industrial concentration framework, where the issue of monopoly is very important.

**Theorem 1 (Link among the properties).**
If a concentration measure satisfies the properties 1 and 6, then it satisfies all the aforementioned six properties.

**Proof.** The outline of the proof is the following. It can be proved that a concentration index satisfying property 1, fulfills also properties 2 and 3. Further, if a concentration measure satisfies properties 1 and 6, then it meets the 4. Finally, properties 2 and 4 imply the 5.

1. Property 1 ⇒ property 3

Let \( s = (s_1, s_2, \ldots, s_n) \) and \( s^* = (s^*_1, s^*_2, \ldots, s^*_n) \) be two portfolios of \( n \) loans each. Let the shares of the two portfolios be such that \( \sum_{i=1}^{n} s_i^* \geq \sum_{i=1}^{n} s_i \) \( \forall k = 1, \ldots, n - 1 \). Let \( \Delta \) denote the difference vector \( s^*_i(s^*_i - s_i) \), that is \( \Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n) = (s^*_1 - s_1, s^*_2 - s_2, \ldots, s^*_n - s_n) \). By construction, \( \sum_{i=1}^{n} \Delta_i \geq 0 \) for \( k = 1, \ldots, n - 1 \) and \( \sum_{i=1}^{n} \Delta_i = 0 \). The idea is to construct, by induction, a sequence of vector shares \( m^1, \ldots, m^t \) such that

- \( m^1 = s(1) \) and \( m^t = s^*(1) \);
- \( m^h \) is obtained from \( m^{h-1} \) by a transfer of part of share of a certain loan to a bigger one.

Suppose that \( m^{h-1} = (m^{h-1}_1, \ldots, m^{h-1}_n) \) is already defined and satisfies the three restrictions:

1) \( m^{h-1}_i \geq 0 \) \( i = 1, \ldots, n; \)
2) $\sum_{i=1}^{n} m_i^{h-1} = 1$;
3) the $m_i^{h-1}$ are decreasingly ranked.

Introduce the vector $\Delta^{h-1} = s^{*} - m^{h-1}$, with $\sum_{i=1}^{n} \Delta_i^{h-1} = 0$. Let $\alpha$ be the first value of the index $i$ in $\{1, \ldots, n\}$ such that $\Delta_i^{h-1} < 0$, that is

$$\alpha = \min_{i \in \{1, \ldots, n\}} \{i : \Delta_i^{h-1} < 0\}.$$ 

By construction, $m_{\alpha}^{h-1} > s_{\{\alpha\}}^{*}$ and $m_{\alpha}^{h-1} = s_{\{\alpha\}}^{*} - \Delta_{\alpha}^{h-1}$. Consider now the loan $\beta$, where $\beta$ is the first value of index $i$ in $\{1, \ldots, \alpha\}$ that realizes the minimum of the strictly positive values of $\Delta_i^{h-1}$:

$$\beta = \min_{i \in \{1, \ldots, \alpha\}} \{\Delta_i^{h-1} : \Delta_i^{h-1} > 0\}.$$ 

Such a $\beta$ always exists because the first non-null component of vector $\Delta^{h-1}$ cannot be negative. By construction, $m_{\beta}^{h-1} < s_{\{\beta\}}^{*}$ and $m_{\beta}^{h-1} = s_{\{\beta\}}^{*} - \Delta_{\beta}^{h-1}$. Since $\beta < \alpha$ and since the components of vector $m^{h-1}$ are decreasingly ordered, $m_{\beta}^{h-1} \geq m_{\alpha}^{h-1}$. By a transfer of the positive quantity $\min(-\Delta_{\alpha}^{h-1}, \Delta_{\beta}^{h-1})$ from the loan $\alpha$ to the loan $\beta$, the vector $\tilde{m}^h$ can be obtained, with shares:

$$\tilde{m}_i^h = \begin{cases} 
  m_{\beta}^{h-1} + \min(-\Delta_{\alpha}^{h-1}, \Delta_{\beta}^{h-1}) & i = \beta \\
  m_{\alpha}^{h-1} - \min(-\Delta_{\alpha}^{h-1}, \Delta_{\beta}^{h-1}) & i = \alpha \\
  m_i^{h-1} & \text{otherwise}
\end{cases}$$

Rearranging the shares $\tilde{m}^h$ in order to have a decreasing order, and denote such vector with $m^h$: that vector is obtained from $m^{h-1}$ by a transfert of positive part of a share of loan $\alpha$ to a bigger share of loan $\beta$. If property 1 holds, then $C_n(m^h) \geq C_n(m^{h-1})$. Following this procedure it is possible to construct by induction a sequence of vector shares: this sequence starts from the portfolio $s$, end at the portfolio $s^*$, increasing the concentration at each step.

The last thing to show is that the number of iteration of the procedure is finite. That can be proved, since at each step, a component of $m^h$ is replaced by the component value of $s^*$: the components of $s^*$ are finite in number, therefore the algorithm converges after a finite number of iterations. Finally it holds that:

$$C_n(s) = C_n(m^1) \leq \ldots \leq C_n(m^{h-1}) \leq C_n(m^h) \leq \ldots C_n(m^l) = C_n(s^*)$$

2. Property 1 $\Rightarrow$ property 2

The proof is of the same type as the previous one. The idea is to construct a sequence of transfers which transforms any portfolio of $n$ loans in the portfolio with equal-amount loans, decreasing the concentration at each step. Thus, the minimum value of the single-name concentration index will be assumed when all the shares have the same value.

3. Properties 1 and 6 $\Rightarrow$ property 4
Let \( s = (s_1, \ldots, s_i, \ldots, s_j, \ldots, s_n) \) be a portfolio of \( n \) loans, and
\[ s^* = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{j-1}, s_m, s_{j+1}, \ldots, s_n) \]
a portfolio obtained by the merge
\[ s_m = s_i + s_j. \]
If \( s_i \leq s_j \) consider the portfolio
\[ s^1 = (s_1, \ldots, s_{i-1}, 0, s_i+1, \ldots, s_{j-1}, s_m, s_{j+1}, \ldots, s_n) \]
achieved by a transfer from a smaller loan \( (s_i) \) to a bigger one \( (s_j) \). Since the property 1 holds, \( C_n(s) \leq C_n(s^1) \).
Now, if the property 6 is assumed, the loan with amount 0 can be removed from \( s^1 \) with no concentration increase. The result is the portfolio \( s^* \) and it holds that
\[ C_n(s) \leq C_n(s^1) \leq C_{n-1}(s^*), \]
therefore property 4 is true.
Then \( C_n(s) \leq C_{n-1}(s^*). \)

4. Properties 2 and 4 \( \Rightarrow \) property 5
Since property 2 is true, \( C_{n+1}(1/n, \ldots, 1/n, 0) \geq C_{n+1}(1/(n+1), \ldots, 1/(n+1)). \)
After a merge, by property 4, it follows that
\[ C_n(1/n, \ldots, 1/n) \geq C_{n+1}(1/n, \ldots, 1/n, 0), \]
and therefore
\[ C_n(1/n, \ldots, 1/n) \geq C_{n+1}(1/(n+1), \ldots, 1/(n+1)). \]
This means that \( C_n(s_{e,n}) \geq C_{n+1}(s_{e,n+1}). \) By iteration, the property 5 holds true.

3 An inequality index: Gini coefficient

The Gini coefficient was introduced by Gini at the beginning of the last century [8]. It is a measure of the “distance” between the equalitarian and the considered situation. In the case of a portfolio of \( n \) loans \( s = (s_1, \ldots, s_n) \) it is defined by
\[
G = \frac{n + 1}{n} - \frac{2}{n - 1} \sum_{i=1}^{n} (n - i + 1) s_{(i)}.
\]

It is a normalized index, since its range is the interval \([0, 1]\): it assumes value 0 if the loans have the same amount, while it equals 1 if the total amount corresponds to an unique loan.

The major drawback is that Gini coefficient does not take into account the portfolio size, therefore it can be considered more an inequality index than a concentration index.

**Theorem 2.** The Gini coefficient \( G \) satisfies the properties: 1, 2, 3 and 5.

**Proof.** Properties 1, 2 and 3
Let \( s = (s_1, \ldots, s_n) \) and \( s^* = (s_1^*, \ldots, s_n^*) \) be two portfolios of \( n \) loans each one, and let
\[ s_{(1)} = (s_{(1)}, s_{(2)}, \ldots, s_{(n)}) \quad \text{and} \quad s_{(1)}^* = (s_{(1)}^*, s_{(2)}^*, \ldots, s_{(n)}^*) \]
be the corresponding ordered portfolios, such that
\[
s_{(k)}^* = \begin{cases} 
  s_{(j)} - h & k = j \\
  s_{(j+1)} + h & k = j + 1 \\
  s_{(k)} & \text{otherwise}
\end{cases}
\]
where
\[
s_j < s_{j+1}, \quad 0 < h < s_{(j+1)} - s_{(j)}, \quad h < s_{(j+2)} - s_{(j+1)}.
\]
Consider the difference between the Gini coefficient of the two portfolios:
\[
G(s^*) - G(s) = G(s^*_i) - G(s_i)
\]
\[
= \frac{n+1}{n-1} - \frac{2}{n-1} \sum_{i=1}^{n} (n-i+1)s^*_i - \frac{n+1}{n-1} - \frac{2}{n-1} \sum_{i=1}^{n} (n-i+1)s_i
\]
\[
= \frac{2}{n-1} \left[ \sum_{i=1}^{n} (n-i+1) \left( s_i - s^*_i \right) \right]
\]
\[
= \frac{2}{n-1} \left[ (n-j+1) \left( s_j - s^*_j \right) + (n-j) \left( s_{j+1} - s^*_{j+1} \right) \right]
\]
\[
= \frac{2}{n-1} \left[ (n-j+1)h + (n-j)(-h) \right]
\]
\[
= \frac{2h}{n-1} > 0.
\]

Then \(G(s^*) > G(s)\). As a generic transfer can be performed with one or more transfers as the considered ones with the restrictions (3), the property 1 holds true.

As the proof of the Theorem 1 shows, property 1 implies properties 2 and 3, which are therefore true for the Gini coefficient.

Property 5
Consider two portfolios with equal-amount loans:

\(s_{e,n} = (1/n, \ldots, 1/n)\) and \(s_{e,m} = (1/m, \ldots, 1/m)\).

By definition of Gini coefficient, it follows that \(G(s_{e,n}) = G(s_{e,m}) = 0\), therefore it holds that \(G_n(s_{e,n}) \leq G_m(s_{e,m})\).

4 Industrial concentration indexes: Hannah-Kay index

For the industrial concentration Hannah and Kay [10] propose the following index (HK)

\[
HK = \left( \sum_{i=1}^{a} s_i^\alpha \right)^{\frac{1}{\alpha}} \quad \alpha > 0 \text{ and } \alpha \neq 1.
\]

On the contrary to the above-mentioned indexes, HK index is inversely proportional to the level of concentration: by increasing concentration the HK index decreases. For this reason we consider the Reciprocal of Hannah-Kay (RHK) index, proposed by Becker, Dullmann and Pisarek [3]
We prefer to analyse RHK and not HK since RHK is proportional to the level of concentration. In particular, if the loans in the portfolio have the same amounts RHK index becomes:

\[ RHK = \left( \sum_{i=1}^{n} s_i^\alpha \right)^{\frac{1}{\alpha-1}} \]

\( \alpha > 0 \) and \( \alpha \neq 1 \).

If the portfolio consists of only one non-null share, RHK is equal to 1. The role of the elasticity parameter \( \alpha \) is to enable us to decide how much weight to attach to the upper portion of the distribution relative to the lower. High \( \alpha \) gives greater weight to the role of the largest firms in the distribution and lower \( \alpha \) emphasizes the presence or absence of small amounts.

**Theorem 3.** The RHK satisfies all the six properties considered in Section 2.

**Proof.** In section 2 it is shown that if the 1 and 6 properties are satisfied, all the six properties of a concentration measure are satisfied.

1. **property**

Let \( s \) and \( s^* \) two portfolios that satisfy the condition (1). The following difference is computed

\[ f(h) = RHK(s^*) - RHK(s) = \left( \sum_{k \neq i,j} s_k^\alpha + (s_j + h)^\alpha + (s_j - h)^\alpha \right)^{\frac{1}{\alpha-1}} - \left( \sum_{k \neq i,j} s_k^\alpha \right)^{\frac{1}{\alpha-1}}. \]

The function \( f(h) \) is continuous for \( h > 0 \) and \( \lim_{h \to 0} f(h) = 0 \). The derivative of \( f(h) \) is computed by obtaining

\[ \frac{\partial f(h)}{\partial h} = \frac{\alpha}{\alpha - 1} \left( \sum_{k \neq i,j} s_k^\alpha + (s_j + h)^\alpha + (s_j - h)^\alpha \right)^{\frac{\alpha}{\alpha-1}} \left[ (s_j + h)^{\alpha-1} - (s_j - h)^{\alpha-1} \right]. \]

In order to determine the sign of this derivative, two cases are considered:

1. \( 0 < \alpha < 1 \)

   In the equation (4) the first and the third factors of the product are negative and the second factor is positive, hence the derivative is positive.

2. \( \alpha \geq 1 \)

   In the equation (4) all the factors are positive, hence the derivative is positive.

6. **property**

Let \( s \) and \( s^* \) two portfolios that satisfy the conditions given in the property 6, the

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1 The proof is similar to the one suggested by Becker, Dullmann and Pisarek [3].
following difference is computed

\[ g(\tilde{x}) = RHK(s^*) - RHK(s) = \left[ \sum_{i=1}^{n} \left( \frac{x_i}{T + \tilde{x}} \right)^{\alpha} + \left( \frac{\tilde{x}}{T + \tilde{x}} \right)^{\alpha} \right]^{\frac{1}{\alpha-1}} - \left[ \sum_{i=1}^{n} s_i^{\alpha} \right]^{\frac{1}{\alpha-1}}. \]

The function \( g(\tilde{x}) \) is continuous for \( \tilde{x} > 0 \) and \( \lim_{\tilde{x} \to 0} g(\tilde{x}) = 0 \), so the entry of a new loan with insignificant exposure \( \tilde{x} \) in the portfolio has insignificant impact on the RHK index.

The derivative of \( g(\tilde{x}) \) is computed by obtaining

\[ \frac{\partial g(\tilde{x})}{\partial \tilde{x}} = \frac{\alpha}{\alpha - 1} \left[ \sum_{i=1}^{n} \left( \frac{x_i}{T + \tilde{x}} \right)^{\alpha} + \left( \frac{\tilde{x}}{T + \tilde{x}} \right)^{\alpha} \right]^{\frac{1}{\alpha-1}} \left( T + \tilde{x} \right)^{-1} \sum_{i=1}^{n} x_i \left( \tilde{x}^{\alpha-1} - x_i^{\alpha-1} \right) \left( T + \tilde{x} \right)^{\alpha+1}. \] (5)

In order to determine the sign of this derivative for \( \tilde{s} < \tilde{s}' \), two cases are considered:

1. \( 0 < \alpha < 1 \)
   In the equation (5) the first factor of the product is negative and the second and the third factors are positive, hence the derivative is negative.
2. \( \alpha \geq 1 \)
   In the equation (4) the first and the second factors are positive and the third factor is negative, hence the derivative is negative.

This means that even if the introduction of the new loan with exposure \( \tilde{x} \) causes a negligible change in RHK, RHK slightly decreases.

Set the equation (5) equal to zero, the limit \( s' \) for \( \tilde{s} \) is obtained

\[ \tilde{s} = \left[ \sum_{i=1}^{n} s_i^{\alpha} \right]^{\frac{1}{\alpha-1}}. \]

It follows that if the new loan has a share \( \tilde{s} \) higher than RHK, the effect of the new loan in reducing the share of the existing large amounts is offset to some extent by the fact that its amount is large.

### 4.1 Herfindahl-Hirschman index

Herfindahl [11] proposes an industrial concentration index whose square root has been proposed by Hirschman [12] (for more details see [13]). For this reason, this index is known as Herfindahl-Hirschman index.

The Herfindahl-Hirschman index (HH) is defined as the sum of squared portfolio shares of all borrowers

\[ HH = \sum_{i=1}^{n} s_i^2 = \frac{\sum_{i=1}^{n} x_i^2}{T^2}. \]
By considering the square of the portfolio share \( s_i \) in the HH index, small exposures affect the level of concentration less than a proportional relationship. When all the credits in the portfolio has the same amounts, the HH index becomes

\[
HH = \sum_{i=1}^{n} \frac{1}{n^2} = \frac{1}{n}.
\]

It is interesting to analyse the relationships among RHK and concentration measures for given value of the elasticity parameter \( \alpha \). If \( \alpha = 2 \) RHK becomes

\[
RHK = \left( \sum_{i=1}^{n} s_i^2 \right)^{\frac{1}{2}} = HH
\]

the HH index.

If a portfolio with only one non-null share is considered, the HH index is equal to 1. The main advantage of the HH index is that it satisfies all six properties of an index of credit concentration, because it is a particular case of the RHK index. Analogously to the Gini coefficient, a drawback of the HH index is that it cannot incorporate the effects of obligor specific credit qualities, e.g. the default probability. Moreover, the HH concentration measures of portfolios with different numbers of borrows \( n \) cannot be compared.

### 4.2 Theil entropy index

Theil [17] proposes an industrial concentration index

\[
TH = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{\bar{x}} \log \frac{x_i}{\bar{x}} = \sum_{i=1}^{n} s_i \log s_i + \log n.
\]

By considering the logarithm of the portfolio share \( s_i \), TH gives relatively more weight to smaller loans.

When all the loans have the same exposure, the TH index is

\[
TH = \log n - \log n = 0.
\]

If a portfolio with only one non-null share is considered, the TH index is equal to \( \log n \).

It is interesting to analyse the relationships among RHK and concentration measures for given value of the elasticity parameter \( \alpha \). The RHK index is undefined for \( \alpha = 1 \) but we can analyse its behaviour when \( \alpha \) is close to 1. Let \( \alpha = 1 + h \), we compute the limit of the RHK for \( h \to 0 \). By applying the Taylor expansion, for \( h \to 0 \) we obtain
Single-name concentration risk

\[ \sum_{i=1}^{n} s_i^{h+1} \simeq \sum_{i=1}^{n} (s_i + h s_i \log s_i) = 1 + h \sum_{i=1}^{n} s_i \log s_i. \]  

(6)

By computing the logarithm of the RHK index and by considering the result (6) we obtain

\[ \lim_{\alpha \to 1} \log RHK = \lim_{h \to 0} \log \left( \sum_{i=1}^{n} (s_i + h s_i \log s_i) \right)^{\frac{1}{h}} = \lim_{h \to 0} \left( \frac{1}{h} \sum_{i=1}^{n} s_i \log s_i \right) = \sum_{i=1}^{n} s_i \log s_i. \]

It is shown later that the previous expression is a transformation of the Theil entropy index (TH) of concentration [17], so it results

\[ \lim_{\alpha \to 1} RHK = \exp(TH - \log n). \]

**Theorem 4.** The TH satisfies the properties 1,2,3,4, and 5.

**Proof.** Property 1 See Cowell (1977).

**Property 4** Consider two portfolios of loans \( \mathbf{s} \) and \( \mathbf{s}^* \) as in property 4. Then:

\[ C_{n-1}(\mathbf{s}^*) - C_n(\mathbf{s}) = s_m \log s_m + \log(n + 1) - [s_i \log s_i + s_j \log s_j + \log n] = \]

\[ = s_m \log s_m - s_i \log s_i - s_j \log s_j + \log(n + 1) - \log n = \]

\[ = \log \left( \frac{s_m}{s_i} \frac{s_m}{s_j} \right) + \log \frac{n + 1}{n} \]

\[ = \log \left( \frac{s_i + s_j}{s_i} \right)^{s_i s_j} \frac{s_i + s_j}{s_j} + \log \frac{n + 1}{n} > 0, \]

(7)

since all the arguments of the logarithms in (7) are greater than 1.

### 4.3 Hall-Tidemann index

Another index, arising from the industrial concentration framework is the Hall-Tidemann index (HT) introduced in [9]. Let \( \mathbf{s} = (s_1, s_2, \ldots, s_n) \) be a portfolio of \( n \) loans, then

\[ HT = \frac{1}{2 \sum_{i=1}^{n} (n - i + 1) s_{(i)} - 1}. \]

This index weights each loan with a value depending on its rank: this feature gives more importance to big loans and to the total number of the loans. If the amounts of all the loans are equal, that is \( \mathbf{s} = (1/n, 1/n, \ldots, 1/n) \), then
It follows that
\[ HT = \frac{1}{2n^{-1} \sum_{i=0}^{n-1} (n-i+1)} - 1 = \frac{1}{n}, \]
while if there is only one loan, the value of the index coincides with 1. A remarkable feature is the link between \( HT \) and the Gini coefficient \( G \): it can be proved that
\[ HT = \frac{1}{n - (n-1)G}. \]

**Theorem 5.** The \( HT \) satisfies all the six properties considered in section 2.

**Proof.** 1 property

Let \( s = (s_1, s_2, \ldots, s_n) \) and \( s^* = (s^*_1, s^*_2, \ldots, s^*_n) \) be two portfolios of \( n \) loans each one, as in property 1. Then, since the Gini coefficient \( G \) satisfies the property 1, it holds \( G(s) \leq G(s^*) \). Hence
\[ G(s) \leq G(s^*) \]
\[ n - (n-1)G(s) \geq n - (n-1)G(s^*) \]
\[ HT(s) \leq HT(s^*). \]

6 property

Let \( s = (s_1, s_2, \ldots, s_n) \) and \( s^* = (s^*_1, s^*_2, \ldots, s^*_n, s^*_{n+1}) \) be two portfolios of \( n \) and \( n+1 \) loans, respectively, as stated in property 6. Consider the difference \( HT(s) - HT(s^*) \):
\[ HT(s) - HT(s^*) = \]
\[ \frac{1}{2[nS(1) + (n-1)x(2) + \ldots + s(n)] - 1} - \frac{1}{2[(n+1)\bar{x} + nS(1) + (n-1)x(2) + \ldots + s(n)] - 1} = \]
\[ \frac{1}{2[nS(1) + (n-1)x(2) + \ldots + s(n)] - 1} - \frac{1}{2[\bar{x} + nS(1) + (n-1)x(2) + \ldots + s(n)] - 1 + 2(n+1)\bar{x}} = \]
\[ \frac{1}{2[nS(1) + (n-1)x(2) + \ldots + s(n)] - 1} - \frac{1}{2[\bar{x} + T(2n+1) - 2[nS(1) + (n-1)x(2) + \ldots + s(n)] - T + \bar{x}(2n+1)]} = \]
\[ c \cdot [T(2n+1) - 2[nS(1) + (n-1)x(2) + \ldots + s(n)] + T] = \]
\[ c \cdot [2nT + 2T - 2[nS(1) + (n-1)x(2) + \ldots + s(n)] > c[2nT + 2T - 2nT] > 0. \]

It follows that \( HT(s) > HT(s^*) \) and therefore property 4 holds true.
5 Numerical applications

For the construction of the most concentrated portfolio, the large exposure limits of the Bank of Italy (Banca d’Italia, 2006) is considered. At first, the exposure of the portfolio $T$ is 1,000 euros. Therefore, the minimum regulatory capital charge of 8% is 80 euros and this is considered the capital requirement of the bank. The Bank of Italy establishes that an exposure is defined as large if it amounts to 10% or more of the bank’s regulatory capital, in this case an exposure is large if it is greater than or equal to 8 euros. According to the Bank of Italy’s regulation, a large exposure must not exceed 25% of the regulatory capital, in this case 20 euros. The sum of all large exposures is limited to eight times the regulatory capital, which corresponds to 640 euros in this case. By considering this regulation, the portfolio with the highest concentration risk $P_1$ consists of 32 exposures equal to 20 euros, 51 equal to 7 euros and one equal to 3 euros. Hence, the total exposure of the portfolio $T_1$ is 1,000 euros and the number of loans is 84.

In order to obtain the portfolio $P_2$, each exposure of 20 euros in the portfolio $P_2$ is divided in two exposure of 10 euros. It follows that the total exposure of the portfolio $T_2$ remains constant ($T = 1,000$) and the number of loans increases ($n = 116$). Moreover, the portfolio $P_3$ is obtained by merging two exposures of 10 euros in one of 20 euros in the portfolio $P_2$. In the portfolio $P_3$, by neglecting the exposure of 20 euros the portfolio $P_4$ is defined. Finally, the last two portfolios are obtained by introducing a medium exposure ($P_5$) and a low exposure ($P_6$) in the portfolio $P_4$. It is important to highlight that both the total exposure $T$ and the number of loans $n$ change in these six portfolios.

\[
\begin{align*}
\text{20...20} & \quad \text{7...7} & \quad \text{3} & \quad T = 1,000 & \quad n = 84 & \quad P_1 \\
\text{32} & \quad \text{51} & \\
\text{10...10} & \quad \text{7...7} & \quad \text{3} & \quad T = 1,000 & \quad n = 116 & \quad P_2 \\
\text{64} & \quad \text{51} & \\
\text{20} & \quad \text{10...10} & \quad \text{7...7} & \quad \text{3} & \quad T = 1,000 & \quad n = 115 & \quad P_3 \\
\text{62} & \quad \text{51} & \\
\text{10...10} & \quad \text{7...7} & \quad \text{3} & \quad T = 980 & \quad n = 114 & \quad P_4 \\
\text{62} & \quad \text{51} & \\
\text{10...10} & \quad \text{7...7} & \quad \text{3} & \quad T = 987 & \quad n = 115 & \quad P_5 \\
\text{62} & \quad \text{52} & \\
\text{10...10} & \quad \text{7...7} & \quad \text{3} & \quad T = 983 & \quad n = 115 & \quad P_5 \\
\text{62} & \quad \text{51} & 
\end{align*}
\]
The portfolio P3 shows a higher single-name concentration of the portfolios P4 and P5, this ordering is satisfied by all the indexes except RHK with \( \alpha = 0.5 \). This result is mainly due to the characteristic of the RHK with \( \alpha = 0.5 \) to stress the importance of smaller exposures.

It is interesting the comparison of the portfolios P4 and P6, even if it falls in the property 6. For RHK with \( \alpha = 0.5 \), \( \alpha = 3 \) and HH, the results in Table 1 are coherent with the proof in Section 4 that these indexes satisfy the property 6 of the irrelevance of small exposures. Moreover, the results in Table 1 show that G and T do not satisfy the property 6, as above mentioned. Finally, for HT the result is only apparently in contrast with the proof in Subsection 4.3. In particular, HT satisfies the property 6 but its superior limit \( \hat{s}' \) of small exposures is lower than 3 (the exposure \( \hat{s} \) added to the portfolio). Indeed, by adding an exposure equal to one to the portfolio, HT decreases.

The comparison between the portfolios P3 and P2 falls in the subadditivity property 4. Even if G does not satisfy this property, in this particular case G satisfies the ordering established by the property 4. The portfolio P5 is obtained by adding a medium exposure to the portfolio P4, for this reason the ordering of the concentration risks of these portfolios is ambiguous. From the portfolio P4 to P5, G and T show an increase of the concentration risk, on the contrary the other indexes a decrease.

All the indexes agree that the concentration risk decreases from P4 to P2. It follows that the impact of the increase of the number of loans in the portfolio is higher than the impact of larger loans. Finally, the portfolio P5 shows a higher concentration risk than that of the portfolio P6. It is important to highlight that only HH satisfies this ordering.

## References