Valuation of R&D Investment Opportunities using the Least-Squares Monte Carlo method

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Abstract

In this paper we show the applicability of the Least Squares Monte Carlo (LSM) in valuing R&D investment opportunities. As it is well known, R&D projects are made in a phased manner, with the commencement of subsequent phase being dependent on the successful completion of the preceding phase. This is known as a sequential investment and therefore R&D projects can be considered as compound options. Moreover, R&D investments often involve considerable cost uncertainty so that they can be viewed as an exchange option, i.e. a swap of an uncertain investment cost for an uncertain gross project value.

In this context, the LSM method is a powerful and flexible tool for capital budgeting decisions and for valuing R&D investments. In fact, this method provides an efficient technique to value complex real investments involving a set of interacting American-type options.

Keyword: Real Options; Exchange Options; Monte Carlo Simulations.
JEL Codes: G13; C15.

1 Introduction

Real options analysis has become a well-know R&D project valuation technique that values managerial flexibility to adjust decisions under uncertainty. For instance, a project that is started now may be abandoned or expanded in the future, making the project conditional. As the investment decision is conditional, it can be regarded as an “option” that is acquired by making the prior investment. The fundamental difference between real options and traditional net present value (NPV) is the flexibility to adapt when circumstances change. Whereas NPV assumes that investments are fixed, an option will be exercised if future opportunities are favourable, otherwise the option will expire without any further cost.

Among the option pricing methods, the Black and Scholes (1973) model is mostly influential. Using financial theory, Myers (1977) was the first to described real options as the opportunities to purchase real assets on possibility favorable terms. In the R&D
real option literature, Thomke (1997) shows empirically that flexibility under uncertainty allows firms to continuously adapt to change and improve products; Hartmann and Hassan (2006) find that real options analysis is used as an auxiliary valuation tool in pharmaceutical investment valuation and so on. As it is well known, R&D projects are made in a phased manner, with the commencement of subsequent phase being dependent on the successful completion of the preceding phase. In this context, Lee and Paxson (2001) view the R&D process and subsequent discoveries as compound exchange options; Andergassen and Sereno (2011) consider $N$-phased investment opportunities where the time evolution of project value follows a jump-diffusion process; Cassimon et al. (2004) provide an analytical model for valuing the phased development of a pharmaceutical R&D project. However, most real investments opportunities are American-type options, to capture the manager flexibility to realize an investment before the maturity time. So Monte Carlo simulation is an attractive tool to solve complex real option model. One of the most new approaches in this environment is the Least-Squares Monte Carlo (LSM) proposed by Longstaff and Schwartz (2001).

Aim of this paper is to value an R&D project through a Compound American Exchange option (CAEO). In particular way we assume that, during the commercialization phase, the firm can realize the respective investment cost before the maturity $T$ benefiting of underlying project value. We value the CAEO applying the LSM method. Moreover, R&D investments often involve considerable cost uncertainty so that they can be viewed as an exchange option, i.e. a swap of an uncertain investment cost for an uncertain gross project value.

The paper is organized as follows. Section 2 analyses the structure of an R&D investment and the valuation of a CAEO using the LSM method. In Section 3 we present four R&D projects that we value using the CAEO. Finally, Section 4 concludes.

2 The Basic Model

In this model, we assume a two-stage R&D investment which structure is the following:

- $R$ is the Research investment spent at initial time $t_0 = 0$;
- $IT$ is the Investment Technology to research innovations paid at time $t_1$. We further suppose that $IT = qD$ is a proportion $q$ of asset $D$, so it follows the same stochastic process of $D$;
- $D$ is the Development investment that the firm needs to invest to receive the R&D project’s value. We assume that $D$ can be realized between $t_1$ and $T$.
- $V$ is the R&D project value.

In particular way, investing $R$ at time $t_0$, the firm obtains a first investment opportunity that can be value as a Compound American Exchange Option (CAEO) denoted by $C(S_k, IT, t_1)$. This option allows to realize the Investment Technology $IT$ at time $t_1$ and to obtain, as underlying asset, the option to realize the market launch; let denote by $S_k(V, D, T - t_1)$ this option value at time $t_1$ with maturity date $T-t_1$ and exercisable $k$ times. In detail, during the market launch, the firm has got a second investment opportunity to invest $D$ between $t_1$ and $T$ and to receive the R&D project value $V$. Specifically, using the LSM model, the firm must decide at any discrete time $\tau_k = t_1 + k\Delta t$, for $k = 1, 2, \cdots h$ with $\tau_h = t_1 + h\Delta t = T$, whether to invest $D$ or to wait, and so to delay the decision at next time. In this way we capture the managerial flexibility to invest $D$ before the maturity $T$ and so to realize the R&D cash flows.
Figure 1 summarizes the R&D investment structure.

\[ C(S_k, T, t_1) \quad S_k(V, D, T-t) \quad V_{t_1} \quad V_T \]

Figure 1: R&D structure.

2.1 Assumptions and general computations

We assume that \( V \) and \( D \) are described by the following stochastic differential equations:

\[
\frac{dV}{V} = (\mu_v - \delta_v)dt + \sigma_v dZ_v^v
\]

(1)

\[
\frac{dD}{D} = (\mu_d - \delta_d)dt + \sigma_d dZ_d^d
\]

(2)

\[
\text{cov} \left( \frac{dV}{V}, \frac{dD}{D} \right) = \rho_{vd} \sigma_v \sigma_d dt
\]

(3)

where \( \mu_v \) and \( \mu_d \) are the expected rates of return, \( \delta_v \) and \( \delta_d \) are the corresponding dividend yields, \( \sigma_v^2 \) and \( \sigma_d^2 \) are the respective variance rates, \( \rho_{vd} \) is the correlation between changes in \( V \) and \( D \). (\( Z_v^v \) and \( Z_d^d \)) are two Brownian processes defined on a filtered probability space \((\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})\).

Assuming that the firm keeps a portfolio of activities which allows it to value activities in a risk-neutral way, the dynamics of the assets \( V \) and \( D \) under the risk-neutral martingale measure \( Q \) are given by:

\[
\frac{dV}{V} = (r - \delta_v)dt + \sigma_v dZ_v^v
\]

(4)

\[
\frac{dD}{D} = (r - \delta_d)dt + \sigma_d dZ_d^d
\]

(5)

\[
\text{cov} \left( \frac{dZ_v^v}{V}, \frac{dZ_d^d}{D} \right) = \rho_{vd} \sigma_v \sigma_d dt
\]

(6)

where \( r \) is the risk-free interest rate, \( Z_v^v \) and \( Z_d^d \) are two Brownian standard motions under the probability \( Q \) with correlation coefficient \( \rho_{vd} \).

Moreover, applying Ito’s lemma and using the logarithm transformation, we get the equation for the price ratio \( P = \frac{V}{D} \) under the risk-neutral probability measure \( Q \):

\[
\frac{dP}{P} = (\mu_v - \delta_v)dt + \sigma_v dZ_v^v - \sigma_d dZ_d^d
\]

(7)

Applying the logarithm transformation for \( D_T \), under the risk-neutral probability measure \( Q \), it results that:

\[
D_T = D_0 \exp \{(r - \delta_d)T\} \cdot \exp \left( -\frac{\sigma_d^2}{2} T + \sigma_d Z_d^d(T) \right)
\]

(8)
where $D_0$ is the value of asset $D$ at initial time.

Since $Z_d^*(T) \sim N(0, \sqrt{T})$, we have that $U \equiv \left( -\frac{\sigma_d^2}{2} T + \sigma_d Z_d^*(T) \right) \sim N \left( -\frac{\sigma_d^2}{2} T, \sigma_d \sqrt{T} \right)$ and therefore $\exp(U)$ is log-normal distributed whose expected value is:

$$E_Q[\exp(U)] = \exp \left( -\frac{\sigma_d^2}{2} T + \frac{\sigma_d^2}{2} T \right) = 1$$

By Girsanov’s theorem, we define a new probability measure $\tilde{Q}$ equivalent to $Q$ whose Radon-Nikodym derivative is:

$$d\tilde{Q} = \exp \left( -\frac{\sigma_d^2}{2} T + \sigma_d Z_d^*(T) \right)$$

(9)

Hence, substituting in (8) we can write:

$$D_T = D_0 e^{(r - \delta_d)T} \frac{d\tilde{Q}}{dQ}$$

(10)

By the Girsanov’s theorem, the process

$$d\tilde{Z}_d = dZ_d^* - \sigma_d dt$$

(11)

is a Brownian motion under the new risk-neutral probability space $(\Omega, A, F, \tilde{Q})$. Therefore, we can write $dZ_d^*$ as

$$dZ_d^* = \rho_{vd} d\tilde{Z}_d + \sqrt{1 - \rho_{cd}^2} dZ'$$

where $Z'$ is a Brownian motion independent of $Z_d^*$ under measure $Q$. Note that $\tilde{Q}$ is defined by (9). Therefore $Z'$ remains a Brownian motion under $\tilde{Q}$ independent of $\tilde{Z}_d$.

Hence, $d\tilde{Z}_d$ defined by:

$$d\tilde{Z}_d = \rho_{vd} d\tilde{Z}_d + \sqrt{1 - \rho_{cd}^2} dZ'$$

(12)

and it is a Brownian motion under $\tilde{Q}$.

By using equations (11) and (12), we can now rewrite the evolution of asset $P$ under the risk-neutral probability $\tilde{Q}$. It results that:

$$\frac{dP}{P} = (\delta_d - \delta_v) dt + \sigma_v d\tilde{Z}_v - \sigma_d d\tilde{Z}_d$$

which we can write as

$$\frac{dP}{P} = (\delta_d - \delta_v) dt + \sigma dZ'$$

(13)

where $(\sigma_v d\tilde{Z}_v - \sigma_d d\tilde{Z}_d) \sim N(0, \sigma \sqrt{dt}) = \sigma dZ'$, $\sigma = \sqrt{\sigma^2_v + \sigma_d^2 - 2\sigma_v \sigma_d \rho_{vd}}$ and $Z'$ is a geometric Brownian motion under $\tilde{Q}$. The logarithm transformation allow us to obtain the risk-neutral price simulation $P'$:

$$P(t) = P_0 \exp \left\{ \left( \delta_d - \delta_v - \frac{\sigma^2}{2} \right) t + \sigma Z'(t) \right\}$$

(14)
2.2 Valuation of Compound American Exchange option using LSM method

The value of CAEO can be determined as the expected value of discounted cash-flows under the risk-neutral probability $Q$:

$$C(S_k, IT, t_1) = e^{-r t_1} E_Q[\max(S_k(V_{t_1}, D_{t_1}, T - t_1) - IT, 0)]$$ (15)

Assuming the asset $D$ as numeraire and using the Eq.(10) we obtain:

$$C(S_k, IT, t_1) = D_0 e^{-\delta d t_1} E_{Q'}[\max(S_k(P_{t_1}, 1, T - t_1) - q, 0)]$$ (16)

where $IT = q D_{t_1}$.

The market launch phase $S_k(P_{t_1}, 1, T - t_1)$ can be analyzed using the LSM method. Like in any American option valuation, the optimal exercise decision at any point in time is obtained as the maximum between immediate exercise value and expected continuation value. The LSM method allows us to estimate the conditional expectation function for each exercise date and so to have a complete specification of the optimal exercise strategy along each path.

The method starts by simulating $n$ price paths of price $P_{t_1}$ using Eq. (14):

$$P_{t_1} = P_0 * \exp(\text{norminv(rand, (dD-dV)*t_1-sigma^2*t_1/2, sigma*sqrt(t_1)))}$$

Let $\hat{P}_{i,t_1}, i = 1 \cdots n$ the simulated prices. Starting from each $i^{th}$ simulated-path, we begin by simulating a discretization of Eq.(14) for $k = 1 \cdots h$:

$$P_t(:,k) = P_t(:,k-1) * \exp((dD-dV-0.5*sigma^2)*dt+sigma*dBt(:,k))$$

where $dBt$ is a random variable with a standard normal distribution. The process is repeated $m$ times over a time horizon $T$. Starting with the last $j^{th}$ price $\hat{P}_{i,j}^T$, $j = 1 \cdots m$, the option value in $T$ can be computed as $S_0(\hat{P}_{i,j}^T, 1, 0) = \max(\hat{P}_{i,j}^T - 1, 0)$:

$$S(:,h) = \max(PPit(:,h)-1,0)$$

Working backward, at time $\tau_{h-1}$, the process is repeated for each $j^{th}$ path. In this case, the expected continuation value may be computed using the analytic expression for an European option $S_1(\hat{P}_{i,j}^{\tau_{h-1}}, 1, \Delta t)$. Moving backwards, at time $\tau_{h-1}$ the firm must decide whether to invest or not. The value of the option is maximized if the immediate exercise exceeds the continuation value, i.e.:

$$\hat{P}_{i,j}^{\tau_{h-1}} - 1 \geq S_1(\hat{P}_{i,j}^{\tau_{h-1}}, 1, \Delta t)$$ (17)

we can find the critical ratio $P_{\tau_{h-1}}^*$ that solve the inequality (17):

$$P_{\tau_{h-1}}^* - 1 = S_1(P_{\tau_{h-1}}^*, 1, \Delta t)$$

and so the condition (17) is satisfied if $\hat{P}_{i,j}^{\tau_{h-1}} \geq P_{\tau_{h-1}}^*$. But it is very heavy to compute the expected continuation value for all previous time and so to determine the critical price $P_{\tau_{h-1}}^*$, $k = 1 \cdots h - 2$, as it is shown in Carr (1995).

The main contribution of the LSM method is to determine the expected continuation values by regressing the subsequent discounted cash flows on a set of basis functions of current state variables. The most common implementation of the LSM is to use the simple powers of state variable $P$.

Let be $L^m$ the basis of functional forms of the state variableable $\hat{P}_{i}^{\tau_{k}}$ that we use as
regressors. We assume that $w = 1 \cdots 3$. At time $\tau_{h-1}$, the least square regression is equivalent to solve the following problem:

$$
\min \sum_{j=1}^{m} \left[ S_0(\hat{P}^{i,j}_t, 1, 0) e^{-r\Delta t} - \sum_{w=1}^{3} a_w L^w(\hat{P}^{i,j}_{\tau_{h-1}}) \right]^2
$$

The optimal $\hat{a} = (\hat{a}_1, \hat{a}_2, \hat{a}_3)$ is then used to estimate the expected continuation value along each path $\hat{P}^{i,j}_{\tau_{h-1}}, j = 1 \cdots m$:

$$
\hat{S}_1(\hat{P}^{i,j}_{\tau_{h-1}}, 1, \Delta t) = \sum_{w=1}^{3} \hat{a}_w L^w(\hat{P}^{i,j}_{\tau_{h-1}})
$$

After that, the optimal decision for each price path is to choose the maximum between the immediate exercise and the expected continuation value. Proceeding recursively until time $t_1$, we have a final vector of continuation values for each price-path $\hat{P}^{i,j}_t$ that allows us to build a stopping rule matrix in Matlab that maximise the value of american option:

```matlab
% Find when the option is exercised:
IStop=find(PPit(:,j-1)-1>=max(XX2*BB,0)) ;
% Find when the option is not exercised:
ICon=setdiff([1:m],IStop) ;
% Replace the payoff function with the value of the option (zeros when
% not exercised and values when exercised):
S(IStop,j-1)=PPit(IStop,j-1)-1;
S(IStop,j:h)=zeros(length(IStop),h-j+1);
S(ICon,j-1)=zeros(length(ICon),1);
```

As consequence, the $i^{th}$ option value approximation $\hat{S}_k^i(\hat{P}^{i}_t, 1, T - t_1)$ can be determined by averaging all discounted cash flows generated by option at each date over all paths $j = 1 \cdots m$.

Finally, it is possible to implement Monte Carlo simulation to approximate the CAEO:

$$
C(S_k, IT, t_1) \approx D_0 e^{-\delta t_1} \left( \sum_{i=1}^{n} \max(S_i^k(\hat{P}^{i}_t, 1, T - t_1) - q, 0) \right)
$$

Appendix A illustrates the full Matlab algorithm to value CAEO. We conclude that, applying real option methodology, the R&D project will be realized at time $t_0$ if $C(S_k, IT, t_1) - R$ is positive, otherwise the investment will be rejected.

### 3 Numerical R&D Applications

Table 1 summarizes the input parameters about four hypothetical R&D investments. The R&D project value $V_0$ is the current value of the underlying project cash flows appropriately discounted. We assume that $V_0$ ranges from 210,000 to 750,000.

For simplicity, we consider a two-staged R&D investment. The projects start with the research phase that is expected to end at time $t_1$ with the discovery of a new good. We consider that $t_1$ = 1 for projects $I$ and $II$, $t_1$ = 2 for $III$ and $t_1$ = 3 for $IV$. On average, this time is about one year for software-technological investments, two years for motor-telecommunication industries and three years for pharmaceutical one.
At time $t_1$, the firm realizes a second investment in technologies to develop the innovation. Its current value is $IT_0$ and we assume that $IT$ is a proportion $q$ of asset $D$.

After that, we have the commercialization phase in which the new product is ready for the market launch. We assume that this phase starts in $t_1$ and ends at time $T$.

After time $T$ each business opportunity disappears. During the commercialization phase, the firm realizes the investment cost $D$ and receives the project value $V$. The investment $D$ can be realized at any time between $t_1$ and $T$ and its current value is $D_0$.

In this way we value the decision flexibilities to capture the R&D cash flows before the maturity $T$. The length of commercialization phase depends by kind of R&D investment: this is shorter for software-technological R&D than motor-pharmaceutical one. So we assume that $T = 2, 3, 5, 7$ for projects I, II, III, IV, respectively.

Appropriately, in order to value the volatility of asset $V$ and $D$, we take into account the quoted shares and traded options of similar companies. Moreover, as an R&D investment presents a high uncertainty about its result, we assume that $\sigma_v$ ranges from 0.54 to 0.88 and $\sigma_d$ from 0.15 to 0.41.

According to financial options, $\delta_v$ denotes the dividends paid on the stock that are foregone by option holder. In real option theory, $\delta_v$ is the opportunity cost of deferring the project and $\delta_d$ is the “dividend yield” on asset $D$.

To compute the value of CAEO we assume that $m = 20000$, $n = 10000$ and the number of discretization per annum $x = 20$. Moreover the Standard Error $\varepsilon_n = \frac{\sigma}{\sqrt{n}}$ is a measure of simulation accuracy and it is estimated as the realised standard deviation of simulations divided by the square root of simulations.

Table 1: Input values for R&D valuation.

<table>
<thead>
<tr>
<th>Project</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R&amp;D$ Project Value</td>
<td>$V_0$</td>
<td>250 000</td>
<td>210 000</td>
<td>750 000</td>
</tr>
<tr>
<td>Development Cost</td>
<td>$D_0$</td>
<td>140 000</td>
<td>200 000</td>
<td>950 000</td>
</tr>
<tr>
<td>Investment Technology</td>
<td>$IT_0$</td>
<td>70 000</td>
<td>120 000</td>
<td>171 000</td>
</tr>
<tr>
<td>Research Investment</td>
<td>$R$</td>
<td>50 000</td>
<td>40 000</td>
<td>35 000</td>
</tr>
<tr>
<td>Exchange Comp. ratio</td>
<td>$q$</td>
<td>0.50</td>
<td>0.60</td>
<td>0.18</td>
</tr>
<tr>
<td>Dividend-Yield of $V$</td>
<td>$\delta_v$</td>
<td>0.20</td>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>Dividend-Yield of $D$</td>
<td>$\delta_d$</td>
<td>0.05</td>
<td>0.05</td>
<td>0</td>
</tr>
<tr>
<td>Time to Maturity</td>
<td>$t_1$</td>
<td>1 year</td>
<td>1 year</td>
<td>2 year</td>
</tr>
<tr>
<td>Time to Maturity</td>
<td>$T$</td>
<td>2 year</td>
<td>3 year</td>
<td>5 year</td>
</tr>
<tr>
<td>Correlation</td>
<td>$\rho_{vd}$</td>
<td>0.38</td>
<td>0.26</td>
<td>0.08</td>
</tr>
<tr>
<td>Volatility of $V$</td>
<td>$\sigma_v$</td>
<td>0.83</td>
<td>0.64</td>
<td>0.54</td>
</tr>
<tr>
<td>Volatility of $D$</td>
<td>$\sigma_d$</td>
<td>0.32</td>
<td>0.41</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 2 contains the Monte Carlo numerical results. In particular way, we have computed four simulated values for each R&D project and then we have considered the average among them to determine the CAEO. For each simulated value we have also computed the standard error (SE). We can observe that this value increase when the
variances $\sigma_v$ and $\sigma_d$ go up. The last two columns of Table 2 shows the comparison between the NPV and the Real option methodology. The NPV is given by the difference between the receipts and expenses in $t_0$, namely $NPV = V_0 - (D_0 + IT_0 + R)$ while the real option value (RO) is the CAEO minus the investment $R$. As we can observe, the NPV of each project is always negative and so, according to the NPV, the firm should reject all projects. On the other hand, considering the real option approach, the investment opportunities I,III, and IV are remuneratives since we take into account both the sequential frame of an R&D, i.e. the possibility that the project may be abandoned in the future, and the managerial flexibility to realize the investment $D$ before the time $T$, and so to benefit of cash flows deriving by R&D efforts.

<table>
<thead>
<tr>
<th>Project</th>
<th>1st MC</th>
<th>2nd MC</th>
<th>3rd MC</th>
<th>4th MC</th>
<th>CAEO</th>
<th>NPV</th>
<th>RO</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>64.444</td>
<td>64.516</td>
<td>64.803</td>
<td>64.592</td>
<td>64.589</td>
<td>-10000</td>
<td>14589</td>
</tr>
<tr>
<td>SE I</td>
<td>0.0111</td>
<td>0.0111</td>
<td>0.0114</td>
<td>0.0113</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SE II</td>
<td>0.0038</td>
<td>0.0038</td>
<td>0.0039</td>
<td>0.0040</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>66.908</td>
<td>65.326</td>
<td>67.973</td>
<td>66.207</td>
<td>66.603</td>
<td>-406000</td>
<td>31603</td>
</tr>
<tr>
<td>SE III</td>
<td>0.0030</td>
<td>0.0030</td>
<td>0.0030</td>
<td>0.0032</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>150910</td>
<td>152120</td>
<td>147410</td>
<td>148353</td>
<td>149698</td>
<td>-46500</td>
<td>49698</td>
</tr>
<tr>
<td>SE IV</td>
<td>0.0329</td>
<td>0.0230</td>
<td>0.0169</td>
<td>0.0240</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### 4 Conclusions

In this paper we have shown how the Least Squares Monte Carlo can be used to evaluate R&D projects. In particular way, an R&D opportunity is a sequential investment and therefore can be considered as a compound option. We have assumed the managerial flexibility to realize the investment $D$ before the maturity $T$ in order to benefit of R&D cash flows. So an R&D project can be view as a Compound American Exchange option, that allows us to couple both the sequential frame and the managerial flexibility of an R&D investment.

Moreover, using the asset $D$ as numéraire, we have reduced the bidimensionality problem of valuing the CAEO to one variable $P$. As we have analyzed, the main contribution of the LSM method is to determine the expected continuation value by regressing the discounted cash flows on the simple powers of variable $P$, and so to overcome the effort to compute the critical prices $P^*_k, k = 1 \cdots h - 2$.

Finally, we have studied four R&D projects. We have observed that the NPV of each project is always negative while, according to real option approach, the investment opportunities I,III and IV are remuneratives.
function LMSCAE0 = LMSCAEO(V,D,q,sigmav,sigmad, rho, dV, dD, t1, T, x, m, n);

sigma = sqrt(sigmav^2 + sigmad^2 - 2*sigmav*sigmad*rho);
P0 = V/D;
d = dV - dD;

for i = 1:n
    Pt = P0*exp(norminv(rand, -d*t1 - sigma^2*t1/2, sigma*sqrt(t1)));
    dBt = sqrt(dt)*randn(m, h); % Brownian motion
    Pt = zeros(m, h); % Initialize matrix
    Pt(:, 1) = Pt1*ones(m, 1); % vector of initial stock price per simulation
    for k = 2:h;
        Pt(:, k) = Pt(:, k-1).*exp((dD-dV-0.5*sigma^2)*dt+sigma*dBt(:, k)); % simulation of prices
    end
    PPit = Pt; % change the name
    % Work Backwards; Initialize CashFlow Matrix
    S = NaN*ones(m, h);
    S(:, h) = max(PPit(:, h)-1, 0);
    for j = h:-1:3;
        % Step 1: Select the path in the money at time j-1
        I = find(PPit(:, j-1) > 1);
        ISize = length(I);
        % Step 2: Project CashFlow at time j onto basis function at time j-1
        if j == h;
            YY = (ones(ISize, 1)*exp(-dD*[1:h-j+1]*dt)).*S(I, j:h);
        else
            YY = sum(((ones(ISize, 1)*exp(-dD*[1:h-j+1]*dt)).*S(I, j:h))');
        end
        PPb = PPit(I, j-1);
        XX = [ones(ISize, 1), PPb, PPb.^2, PPb.^3];
        BB = pinv(XX'*XX)*XX'*YY;
        XX2 = [ones(m, 1), PPb2, PPb2.^2, PPb2.^3];
        % Find when the option is exercised:
        IStop = find(PPit(:, j-1) > max(XX2*BB, 0)) ;
        % Find when the option is not exercised:
        ICon = setdiff([1:m], IStop);
        % Replace the payoff function with the value of the option:
        S(IStop, j-1) = PPit(ISStop, j-1)-1;
        S(IStop, j:h) = zeros(length(ISStop), h-j+1);
        S(ICon, j-1) = zeros(length(ICon), 1);
    end
    YY = sum(((ones(m, 1)*exp(-dD*[1:h-1]*dt)).*S(:, 2:h))');
    AEOSim(i) = mean(YY);
    PAYOFF(i) = max(AEOSim(i) - q, 0);
end
CAEO = D*exp(-dD*t1)*mean(PAYOFF);
References


